Math 211

Lecture #12

Numerical Methods — Euler's Method

September 23, 2002

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- It is a discrete approximation to a solution.
- We make an error on purpose to enable us to compute an approximation.
- Extremely important to understand the size of the error.

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- and values $y_0, y_1, y_2, \ldots, y_{N-1}, y_N$
  with y_j approximately equal to $y(t_j)$.
- Making an error $E_j = y(t_j) - y_j$.

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Types of Solvers

- We will discuss and use four solvers
  - Euler's method,
  - second order Runge-Kutta,
  - fourth order Runge-Kutta,
  - and `ode45`.
- Everything works for first order systems almost without change.

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etc.

MATLAB routine `eulerdemo.m`

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- Demonstrates how truncation error can propagate exponentially.

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- Demonstrates how the total error is the sum of propagated truncation errors.

Error Analysis – First Step

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- $y(t_1) - y_1 = R(h)$

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- $y(t_1) - y_1 = R(h)$
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where C & L are constants that depend on f.

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- Good news: the error decreases as h decreases.
- Bad news: the error can get exponentially large as the length of the interval [i.e., b-a] increases.

MATLAB routine `eul.m`

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- t_0 - initial time; t_f - final time.

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- Example: $y' = y^2 - t$
- Derivative m-file:

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function ypr = george(t,y)

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- Save as `george.m`.

Use of `eul.m`

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- Solve $y' = y^2 - t$.
- Use the derivative m-file `george.m`.

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- Solve $y' = y^2 - t$.
- Use the derivative m-file `george.m`.
- Use $t_0 = 0$, $t_f = 10$, $y_0 = 0.5$, and several step sizes.

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Use of `eul.m`

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- Use the derivative m-file `george.m`.
- Use $t_0 = 0$, $t_f = 10$, $y_0 = 0.5$, and several step sizes.
- Syntax: `[t,y] = eul('george',[0,10],0.5,h);`

Experimental Error Analysis

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- IVP $y' = \cos(t)/(2y-2)$ with $y(0) = 3$
- Exact solution: $y(t) = 1 + \sqrt{4 + \sin t}$.
- Solve using Euler's method and compare with the exact solution.
- Do this for several step sizes.

Derivative m-file `ben.m`

```
function yprime = ben(t,y)

yprime = cos(t)/(2*y-2);
```

M-file batch.m

```
[teuler,yeuler]=eul('ben',[0,3],3,h);
t=0:0.05:3;
y=1+sqrt(4+sin(t));
plot(t,y,teuler,yeuler,'o')
legend('Exact','Euler')
shg
z=1+sqrt(4+sin(teuler));
maxerror=max(abs(z-yeuler))
```