Math 211

Lecture #12

Numerical Methods — Euler's Method

September 23, 2002



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- We make an error on purpose to enable us to compute an approximation.
- Extremely important to understand the size of the error.

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Return

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- and values $y_0, y_1, y_2, \ldots, y_{N-1}, y_N$ with y_j approximately equal to $y(t_j)$.
- Making an error $E_j = y(t_j) y_j$.

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 - Euler's method,
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- Everything works for first order systems almost without change.

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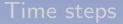
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$$t_N = a + Nh = b$$





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 - Set $y_1 = y_0 + f(t_0, y_0)h$, so $y(t_1) \approx y_1$.





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Input t_0 and y_0 .



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Thus,

 $y_1 = y_0 + f(t_0, y_0)h$ and $t_1 = t_0 + h$

Return

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Numerical methods



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Numerical methods

Time steps

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etc.

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Numerical methods

Time steps

• Demonstrates truncation error.



- Demonstrates truncation error.
- Demonstrates how truncation error can propagate exponentially.

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- Demonstrates how truncation error can propagate exponentially.
- Demonstrates how the total error is the sum of propagated truncation errors.

• Euler's approximation



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 $\overline{y(t_1)} = y(t_0 + h)$

• Euler's approximation

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• Taylor's theorem

 $y(t_1) = y(t_0 + h) = y(t_0) + y'(t_0)h + R(h)$ $|R(h)| \le Ch^2$



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Return

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• $y(\overline{t_1}) - y_1 = \overline{R(h)}$



• Euler's approximation

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- $y(t_1) y_1 = R(h)$
- The truncation error at each step is the same as the Taylor remainder, and $|R(h)| \leq Ch^2$.

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Error Analysis

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Maximum error
$$\leq C\left(e^{L(b-a)}-1\right)h$$
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where C & L are constants that depend on f.

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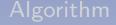
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- Good news: the error decreases as h decreases.
- Bad news: the error can get exponentially large as the length of the interval [i.e., b-a] increases.



Syntax:



MATLAB routine eul.m Syntax: [t,y] = eul(derfile, $[t_0, t_f], y_0, h$);

Algorithm

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Syntax: [t,y] = eul(derfile,[t₀,t_f],y₀,h);
 derfile - derivative m-file defining the equation.

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- *h* step size.

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- Example: $y' = y^2 t$
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• Example:
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• Derivative m-file:

function ypr = george(t,y)
ypr = y^2 - t;

The derivative m-file describes the differential equation.

• Example:
$$y' = y^2 - t$$

• Derivative m-file:

• Save as george.m.

• Solve $y' = y^2 - t$.

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$$y' = y^2 - t$$
.

• Use the derivative m-file george.m.

- Solve $y' = y^2 t$.
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- Solve $y' = y^2 t$.
- Use the derivative m-file george.m.
- Use $t_0 = 0$, $t_f = 10$, $y_0 = 0.5$, and several step sizes.
- Syntax: [t,y] = eul('george',[0,10],0.5,h);

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- Exact solution: $y(t) = 1 + \sqrt{4 + \sin t}$.
- Solve using Euler's method and compare with the exact solution.
- Do this for several step sizes.

Derivative m-file ben.m

function yprime = ben(t,y)

yprime = $\cos(t)/(2*y-2)$;

Experimental analysis

M-file batch.m

```
[teuler,yeuler]=eul('ben',[0,3],3,h);
t=0:0.05:3;
y=1+sqrt(4+sin(t));
plot(t,y,teuler,yeuler,'o')
legend('Exact','Euler')
shg
z=1+sqrt(4+sin(teuler));
maxerror=max(abs(z-yeuler))
```