

Math 211

Lecture #16

Geometry of Solution Sets

October 2, 2002

Systems and Solutionos

- Consider the system

$$\begin{aligned} 3x + 4y - 5z &= 3 \\ -2x + 3z &= -7 \end{aligned} \Leftrightarrow \mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & -5 \\ -2 & 0 & 3 \end{pmatrix},$$
$$\mathbf{x} = (x, y, z)^T, \quad \mathbf{b} = (3, -7)^T.$$

- A *solution* is a vector \mathbf{x} such that $\mathbf{Ax} = \mathbf{b}$.
 $(2, -2, -1)^T$ is a solution.

Solution Sets

- The *solution set* for a system of equations is set of *all solutions* .
 - ◆ What kinds of sets can be solution sets?
 - ◆ Can a circle be the solution set for a system of linear equations?
- We will find all possibilities in 2 and 3 dimensions.
- We will use geometry to find the answer.

One Equation in Two Variables

Example: $2x - 3y = 1$

- The **solution set** is a line in the plane.
- The solution set consists of all vectors of the form

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x \\ (-1 + 2x)/3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1/3 \end{pmatrix} + x \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}\end{aligned}$$

- x is a free parameter, $-\infty < x < \infty$.

Parametric Equation for a Line

$$\mathbf{u} = \mathbf{u}_0 + x\mathbf{v}, \quad -\infty < x < \infty$$

- In our case $\mathbf{u}_0 = (0, -1/3)^T$ and $\mathbf{v} = (1, 2/3)^T$
- The vector \mathbf{u}_0 locates one point on the line.
- The vector \mathbf{v} gives the direction of the line.
- The number x tells how far the point \mathbf{u} is from \mathbf{u}_0 .

Two Equations in Two Variables

Example:

$$\begin{array}{l} 2x - 3y = 1 \\ x + y = 3 \end{array}, \quad \text{or} \quad \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- Two equations \Rightarrow two lines.
- There are three possibilities:
 - ♦ Two lines can intersect in one point. $(2, 1)^T$
 - ♦ Two lines can be the same, and intersect in a line.
 - ♦ Two lines can be parallel. The intersection is empty. Such systems are *inconsistent*.

Possible Solution Sets in Dimension 2

There are precisely three possibilities:

- A **line**.
- A **single point**.
- The **empty set**. The system is inconsistent.

Can a circle be the solution set for a system of linear equations?

One Equation in Three Variables

Example: $2x - 3y + 4z = 1$

- The solution set is a plane in 3-space.
- The solution set is all vectors of the form

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x \\ y \\ (1 - 2x + 3y)/4 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + x \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 3/4 \end{pmatrix} \end{aligned}$$

where x and y are free parameters ($-\infty < x, y < \infty$).

Parametric Equation for a Plane

$$\mathbf{u} = \mathbf{u}_0 + x\mathbf{v} + y\mathbf{w}, \quad -\infty < x, y < \infty$$

- In **the example** $\mathbf{u}_0 = (0, 0, 1/4)^T$,
 $\mathbf{v} = (1, 0, -1/2)^T$, and $\mathbf{w} = (0, 1, 3/4)^T$
- \mathbf{u}_0 locates one point on the plane.
- \mathbf{v} and \mathbf{w} give two different directions in the plane.
- \mathbf{u} differs from \mathbf{u}_0 by the linear combination of \mathbf{v} and \mathbf{w} with coefficients x & y .

Two Equations in Three Variables

Example:

$$\begin{aligned} 2x - 3y + 4z &= 1 \\ x + y - z &= 3 \end{aligned} \quad \text{or} \quad \begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- The solution set consists of all vectors of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 13 - 6x \\ 10 - 5x \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \\ 10 \end{pmatrix} + x \begin{pmatrix} 1 \\ -6 \\ -5 \end{pmatrix}$$

- This is a **line** in 3-space.

Three Equations in Three Variables

Example:

$$\begin{aligned} 2x - 3y + 4z &= 1 \\ x + y - z &= 3 \\ 3x - y + 3z &= 5 \end{aligned} \quad \text{or} \quad \begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

- The only solution is $(x, y, z)^T = (2, 1, 0)^T$.

Solution Sets in Dimension 3

- Two equations \Rightarrow two planes
 - ◆ There are three possibilities — \emptyset , a line, or a plane.
- Three equations \Rightarrow three planes
 - ◆ There are four possibilities — \emptyset , a point, a line, or a plane.
 - ◆ In **the example** the three planes intersect in one point.

Possible Solution Sets in Dimension 3

There are precisely **four possibilities**:

- A plane.
- A line.
- A single point.
- The empty set. The system is inconsistent.

Can a sphere be a solution set of a system of linear equations?

Solution Sets in Higher Dimension

By analogy with dimensions 2 & 3, we expect the following:

- The solution set could be \emptyset if the system is inconsistent.
- The solution set could be a point.
- If a solution set contains 2 points, then it contains the **line** through them.
- If a solution set contains 3 points not on the same line, then it contains the **plane** through them.

Solution Sets of Homogeneous Equations

- $\mathbf{0}$ is the vector with all entries $= 0$. $\mathbf{0}$ is referred to as the *zero vector* or the *origin*.
- A *homogeneous system* is one of the form $A\mathbf{x} = \mathbf{0}$.
- A homogeneous system always has $\mathbf{0}$ as a solution.
- Hence the *solution set* of a homogeneous system is never the empty set.