Math 211

Lecture #17

Solving Systems of Equations

October 4, 2002

Solving Systems of Equations

- We want to find a way to find the solution set of any system.
- We will build towards the method by looking at a series of examples.
- We will start by solving a 2×2 system using three different, but closely related methods.

Example 1

$$x + y = 3$$

$$2x - 3y = 1$$

- **Method 1:** Solve the first equation for x and substitute into the second equation.
- **Method 2:** Add -2 times the first equation to the second equation to eliminate x.
- **Method 3:** Use the augmented matrix and add -2 times the first row to the second row.

Comparison

- Using any of these methods we get the simpler system

$$\begin{aligned} x + y &= 3 \\ -5y &= -5 \end{aligned}$$

- The simple system has the same solutions as the original system .
- The simple system is very easy to solve
  - Solve the last equation first, $y = 1$.
  - Then the first equation, $x = 2$.
  - This is called *backsolving*.

Return

Method of Solution

The method is called *elimination and backsolving*, or *Gaussian elimination.* There are four steps:

1. Write down the augmented matrix.
2. Eliminate as many coefficients as possible.
   - This is not well defined yet.
3. Write down the simplified system.
4. Solve the simplified system by backsolving.

Example 2

$$y - 2z = -1$$

$$2x - 3y + 4z = 5$$

$$-2z + 5y - 8z = -7$$

1. Write down the augmented matrix.

$$\begin{pmatrix} 0 & 1 & -2 & -1 \\ 2 & -3 & 4 & 5 \\ -2 & 5 & -8 & -7 \end{pmatrix}$$

Example 2 — Elimination

2. Eliminate as many coefficients as possible.

- Interchange rows 1 and 2. $R_1 \leftrightarrow R_2$
- $R_3 \rightarrow R_3 + R_1$.
- $R_3 \rightarrow R_3 + (-2) \cdot R_2$. The result is

$$\begin{pmatrix} 2 & -3 & 4 & 5 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 2 — Backsolving

3. Write down the simplified system.

$$\begin{aligned} 2x - 3y + 4z &= 5 \\ y - 2z &= -1 \end{aligned}$$

4. Solve the simplified system by backsolving.
   - z is a free variable. Set $z = t$.
   - $y = -1 + 2t$.
   - $x = 1 + t$.

Example 2 — Solution Set

The solutions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

- This is a line in $\mathbf{R}^3$.
  - $\mathbf{R}^n$ is the set of all n-vectors.

Elimination — Equations

We only use operations on the equations which will lead to systems of equations with the same solutions. These are:

- Add a multiple of one equation to another.
- Interchange two equations.
- Multiply an equation by a non-zero number.

Elimination — Row operations

The corresponding operations on the rows of the augmented matrix are called *row operations*.

- Add a multiple of one row to another.
- Interchange two rows.
- Multiply a row by a non-zero number.

The Goal of Elimination

- How simple can we make the augmented matrix?

$$\begin{pmatrix} P & * & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * & * \\ 0 & 0 & 0 & P & * & * & * & * & * \\ 0 & 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & P & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- P is a nonzero number, $*$ is any number.

Row Echelon Form

- The *pivot* of a row in a matrix is the first non-zero element from the left.
- A matrix is in *row echelon form* if every pivot lies strictly to the right of those in rows above.
- Any matrix can be reduced to row echelon form using the first two of the row operations.
- When an augmented matrix has been reduced to row echelon form, the corresponding system can be easily solved by backsolving.

Reduced Row Echelon Form

- Row echelon form, plus all pivots $= 1$ and all other entries in a pivot column are 0.

$$\begin{pmatrix} 1 & 0 & * & 0 & 0 & * & 0 & 0 & * \\ 0 & 1 & * & 0 & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 3

$$
\begin{aligned}
3x_2 - 4x_3 &= -7 \\
-x_1 + 2x_2 &= -3 \\
3x_1 + 2x_2 + x_3 &= 2
\end{aligned}
$$

1. The augmented matrix is

$$
\begin{pmatrix}
0 & 3 & -4 & -7 \\
-1 & 2 & 0 & -3 \\
3 & 2 & 1 & 2
\end{pmatrix}
$$

Example 3 — Elimination

2. Elimination:

   - $R_1 \leftrightarrow R_2$.
   - $R_3 \to R_3 + 3 \cdot R_1$.
   - $R_3 \to R_3 + (-\frac{8}{3}) \cdot R_2$.
   - $R_3 \to \frac{3}{35} \cdot R_3$.

- We get

$$\begin{pmatrix} -1 & 2 & 0 & -3 \\ 0 & 3 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Example 3 — Back Solving

3. The simplified system is

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 3x_2 - 4x_3 &= -7 \\ x_3 &= 1 \end{aligned}$$

4. Backsolve: $x_3 = 1$, $x_2 = -1$, and $x_1 = 1$.

Elimination using MATLAB

- $R_i \longrightarrow R_i + aR_j$
  - `>> M(i,:)  = M(i,:)  + a*M(j,:)`
- $R_i \leftrightarrow R_j$
  - `>> M([i,j],:)  = M([j,i],:)`
- $R_i \longrightarrow aR_i$
  - `>> M(i,:)  = a*M(i,:)`

Example 4 $A\mathbf{x} = \mathbf{b}$

$$A = \begin{pmatrix} 1 & 2 & 5 & -1 \\ 1 & 2 & -3 & 8 \\ 3 & 6 & 7 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ -12 \\ -16 \end{pmatrix}$$

1. Augmented matrix:

$$M = [A,\ \mathbf{b}] = \begin{pmatrix} 1 & 2 & 5 & -1 & -2 \\ 1 & 2 & -3 & 8 & -12 \\ 3 & 6 & 7 & 6 & -16 \end{pmatrix}$$

.

Example 4 — Elimination

2. Elimination using MATLAB yields.

$$\begin{pmatrix} 1 & 2 & 5 & -1 & -2 \\ 0 & 0 & -8 & 9 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- The yellow entries are the pivots.

3. Simplified system:

$$\begin{aligned} x_1 + 2x_2 + 5x_3 - x_4 &= -2 \\ -8x_3 + 9x_4 &= -10 \end{aligned}$$

4. Backsolve:

- There are pivots in columns 1 & 3. These are *pivot columns*. x_1 and x_3 are called *pivot variables*.
- The other columns are called *free columns*. The variables x_2 and x_4 are called *free variables*.

- The free variables may be assigned arbitrary values: $x_2 = s$ and $x_4 = t$.
- Backsolve for the pivot variables.

$$x_3 = (10 + 9x_4)/8 = 5/4 + 9t/8$$

$$\begin{aligned} x_1 &= -2 - 2x_2 - 5x_3 + x_4 \\ &= -2 - 2s - 5(5/4 + 9t/8) + t \\ &= -33/4 - 2s - 37t/8 \end{aligned}$$

- The solutions are the vectors

$$\mathbf{x} = \begin{pmatrix} -33/4 \\ 0 \\ 5/4 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -37/8 \\ 0 \\ 9/8 \\ 0 \end{pmatrix}$$

- The solution set is a plane in $\mathbf{R}^4$.

Method of Solution for $A\mathbf{x} = \mathbf{b}$

There are four steps:

1. Use the augmented matrix $M = [A,\ \mathbf{b}]$.
2. Use row operations to reduce the augmented matrix to row echelon form.
3. Write down the simplified system.
4. Backsolve.
   - Assign arbitrary values to the free variables.
   - Backsolve for the pivot variables.