Math 211

Lecture #19

Nullspaces and Subspaces

October 9, 2002

Homogeneous Systems

A homogeneous system has the form $A\mathbf{x} = \mathbf{0}$.

- The augmented matrix $M = [A,\ \mathbf{0}]$ has all zeros in the last column.
- During elimination the column of zeros is unchanged.
  - It is not really necessary to augment a homogeneous system.
- A homogeneous system is always consistent.
  - The zero vector $\mathbf{0}$ is always a solution.

Return Solution method

Homogeneous Systems (cont.)

- When does a homogeneous system have a nonzero solution?
  - A homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nonzero solution if and only if the row echelon form of A has a free column.
- A homogeneous system of n equations and m unknowns with $n < m$ always has a nonzero solution.

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Example 1

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & 5 \end{pmatrix}$$

- The solution set to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t(-3, -2, 1)^T$ for $-\infty < t < \infty$.
  - There are nonzero solutions because the reduced echelon form of A has a free column.
- The system $A\mathbf{x} = (1, 1, 0)^T$ has no solutions.
  - The reduced echelon form of $M = [A, b]$ has a pivot in the last column.

Example 2

$$B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & 4 \end{pmatrix}$$

- The reduced echelon form of B has no free columns.
- The solution set for the homogeneous system $B\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
- The inhomogeneous system $B\mathbf{x} = \mathbf{b}$ has a unique solution for any vector $\mathbf{b}$.

Square Homogenous Systems

Suppose that A is a square $(n \times n)$ matrix.

- If the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nonzero solution then there are vectors $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ has no solutions.
- If the homogeneous system $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$.

Singular and Nonsingular Matrices

The $n \times n$ matrix A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any choice of right hand side $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is nonsingular if and only if the row echelon form of A has only nonzero entries along the diagonal.

- A is nonsingular $\Leftrightarrow$ the reduced row echelon form of A is the identity matrix I.

Return Example 1 Example 2

Properties of Singular and Nonsingular Matrices

Proposition: If the $n \times n$ matrix A is nonsingular then the equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for any right hand side $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.

- This is a result that we will use repeatedly.

Return Example 1 Example 2

Invertible Matrices

An $n \times n$ matrix A is *invertible* if there is an $n \times n$ matrix B such that $AB = BA = I$. The matrix B is called an *inverse* of A.

- If B_1 and B_2 are both inverses of A, then $B_1 = B_2$.
- The inverse of A is denoted by A^{-1}.
- Invertible $\Rightarrow$ nonsingular.
- Nonsingular $\Rightarrow$ invertible.

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Computing the inverse A^{-1}

- Form the matrix $[A,\ I]$.
- Do elimination until the matrix is in reduced row echelon form.
  - If A is invertible this will have the form $[I,\ B]$.
- Then $A^{-1} = B$.
- Examples: $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & 4 \end{pmatrix}$
- In MATLAB use `inv`.

The Solution Set of $A\mathbf{x} = \mathbf{b}$

- Example: The matrix A from Example 1 with $\mathbf{b} = (7, 13, 8)^T$. $\mathbf{x} = (7, 1, 0)^T + t(-3, -2, 1)^T$.

Theorem: Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector $\mathbf{x}_h$ such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

- Thus, the solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution $\mathbf{x}_p$ and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.

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Solution Set of a Homogeneous System

- The solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the *nullspace* of the matrix A. It is denoted by $\mathrm{null}(A)$. Thus

$$\mathrm{null}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}.$$

- What are the properties of nullspaces?
- Is there a convenient way to describe them?

Return Solution set

Properties of Nullspaces

Proposition: Let A be a matrix.

1. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathrm{null}(A)$, then $\mathbf{x} + \mathbf{y}$ is in $\mathrm{null}(A)$.
2. If a is a scalar and $\mathbf{x}$ is in $\mathrm{null}(A)$, then $a\mathbf{x}$ is in $\mathrm{null}(A)$.

Definition: A nonempty subset V of $\mathbf{R}^n$ that has the properties

1. if $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, $\mathbf{x} + \mathbf{y}$ is in V,
2. if a is a scalar, and $\mathbf{x}$ is in V, then $a\mathbf{x}$ is in V,

is called a *subspace* of $\mathbf{R}^n$.

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Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through $\mathbf{0}$, $V = \{t\mathbf{v} \mid t \in \mathbf{R}\}$, is a subspace.
- A plane through $\mathbf{0}$, $V = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\}$, is a subspace.
- $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are subspaces of $\mathbf{R}^n$.
    - These are called the *trivial subspaces.*

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Linear Combinations

Proposition: Any linear combination of vectors in a subspace V is also in V.

- Subspaces of $\mathbf{R}^n$ have the same linear structure as $\mathbf{R}^n$ itself.
- The nullspace of a matrix is a subspace, so it has the same linear structure as $\mathbf{R}^n$.

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Example

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

The nullspace of A is the set

$$\text{null}(A) = \{a\mathbf{v} \mid a \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1)^T$.

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Example

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

The nullspace of B is the set

$$\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$.

- $\text{null}(B)$ consists of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$.

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The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …, and $\mathbf{v}_k$ is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

Proposition: If $\mathbf{v}_1$, $\mathbf{v}_2$, …, and $\mathbf{v}_k$ are all vectors in $\mathbf{R}^n$, then $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbf{R}^n$.

Return $\text{null}(A)$ $\text{null}(B)$ Examples

How do we know if $\mathbf{w}$ is in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$?

1. Form the matrix $V = [\mathbf{v}_1, \mathbf{v}_2, \cdots . \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, and $\mathbf{v}_k$ as its columns.
2. Solve the system $V\mathbf{a} = \mathbf{w}$.
   a. If there are no solutions, $\mathbf{w}$ is *NOT* in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$?
   b. If there is a solution $\mathbf{a} = (a_1, a_2, \cdots, a_k)^T$, then

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k.$$

Examples

Let $\mathbf{v}_1 = (1,\, 2)^T$, $\mathbf{v}_2 = (1,\, 0)^T$, and $\mathbf{v}_3 = (2,\, 0)^T$.

- $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{R}^2$. (Proof?)
- $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_3) = \mathbf{R}^2$. (Proof?)
- $\mathrm{span}(\mathbf{v}_2, \mathbf{v}_3) = \mathrm{span}(\mathbf{v}_2)$. (Proof?)
  - $\mathrm{span}(\mathbf{v}_2, \mathbf{v}_3) = \{t\mathbf{v}_2 \,|\, t \in \mathbf{R}\}$.
  - $\mathbf{v}_2$ and $\mathbf{v}_3$ have the same direction.

Row operations

The permissable operations on the rows of the augmented matrix are called *row operations*.

- Add a multiple of one row to another.
- Interchange two rows.
- Multiply a row by a non-zero number.

Row Echelon Form

A matrix is in *row echelon form* if every pivot lies strictly to the right of those in rows above.

$$\begin{pmatrix} P & * & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * & * \\ 0 & 0 & 0 & P & * & * & * & * & * \\ 0 & 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & P & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- P is a pivot, $*$ is any number.

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Method of Solution for $A\mathbf{x} = \mathbf{b}$

The method is called *elimination and backsolving*, or *Gaussian elimination*. There are four steps:

1. Use the augmented matrix $M = [A,\ \mathbf{b}]$.
2. Use row operations to reduce the augmented matrix to row echelon form.
3. Write down the simplified system.
4. Backsolve.
   - Assign arbitrary values to the free variables.
   - Backsolve for the pivot variables.

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