

Math 211

Lecture #20

Nullspaces and Subspaces

October 11, 2002

The Solution Set of $A\mathbf{x} = \mathbf{b}$

Theorem: Let \mathbf{x}_p be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
 2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector \mathbf{x}_h such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.
- Thus, the solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution \mathbf{x}_p and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.

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The Nullspace of a Matrix

- The solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the *nullspace* of the matrix A . It is denoted by $\text{null}(A)$. Thus

$$\text{null}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}.$$

- What are the properties of nullspaces?
- Is there a convenient way to describe them?

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Properties of Nullspaces

Proposition: Let A be a matrix.

1. If \mathbf{x} and \mathbf{y} are in $\text{null}(A)$, then $\mathbf{x} + \mathbf{y}$ is in $\text{null}(A)$.
2. If a is a scalar and \mathbf{x} is in $\text{null}(A)$, then $a\mathbf{x}$ is in $\text{null}(A)$.

Definition: A nonempty subset V of \mathbf{R}^n that has the properties

1. if \mathbf{x} and \mathbf{y} are vectors in V , then $\mathbf{x} + \mathbf{y}$ is in V ,
2. if a is a scalar, and \mathbf{x} is in V , then $a\mathbf{x}$ is in V ,

is called a *subspace* of \mathbf{R}^n .

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Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through $\mathbf{0}$, $V = \{t\mathbf{v} \mid t \in \mathbf{R}\}$, is a subspace.
- A plane through $\mathbf{0}$, $V = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\}$, is a subspace.
- $\{\mathbf{0}\}$ and \mathbf{R}^n are subspaces of \mathbf{R}^n .
 - ♦ These are called the *trivial subspaces*.

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Linear Combinations

Proposition: Any linear combination of vectors in a subspace V is also in V .

- Subspaces of \mathbf{R}^n have the same linear structure as \mathbf{R}^n itself.
- The nullspace of a matrix is a subspace, so it has the same linear structure as \mathbf{R}^n .
- The product of a matrix A and a vector \mathbf{x} is the linear combination of the column vectors in A with the elements of \mathbf{x} as coefficients.

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Another Example of a Nullspace

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nullspace of A is the set

$$\text{null}(A) = \{a\mathbf{v} \mid a \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1)^T$.

- The nullspace of A consists of all multiples of \mathbf{v} .

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Another Example of a Nullspace

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nullspace of B is the set

$$\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$.

- $\text{null}(B)$ consists of all linear combinations of \mathbf{v} and \mathbf{w} .

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The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).$$

Proposition: If $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k are all vectors in \mathbf{R}^n , then $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ is a subspace of \mathbf{R}^n .

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[null\(A\)](#)

[null\(B\)](#)

[Examples](#)

How Do We Know if $\mathbf{w} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k as its columns.
2. Solve the system $M\mathbf{a} = \mathbf{w}$.
 - a. If there are no solutions, \mathbf{w} is *NOT* in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$.
 - b. If there is a solution $\mathbf{a} = (a_1, a_2, \dots, a_k)^T$, then

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k$$

is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$.

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Span

Matrix product

Examples of Spans

Let $\mathbf{v}_1 = (1, 2)^T$, $\mathbf{v}_2 = (1, 0)^T$, and $\mathbf{v}_3 = (2, 0)^T$.

- $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{R}^2$. (Proof?)
- $\text{span}(\mathbf{v}_1, \mathbf{v}_3) = \mathbf{R}^2$. (Proof?)
- $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{v}_2)$. (Proof?)
 - ♦ $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \{t\mathbf{v}_2 \mid t \in \mathbf{R}\}$.
 - ♦ \mathbf{v}_2 and \mathbf{v}_3 have the same direction.

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Span

Method

Linear Independence of Two Vectors

We need a condition that will keep unneeded vectors out of a spanning list. We will work toward a general definition.

- Two vectors are *linearly dependent* if one is a scalar multiple of the other.
 - ♦ \mathbf{v}_2 and \mathbf{v}_3 are linearly dependent.
 - ♦ \mathbf{v}_1 and \mathbf{v}_2 are *linearly independent*.

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Linear Independence of Three Vectors

- Three vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are *linearly dependent* if one is a linear combination of the other two.
 - ♦ Example: $\mathbf{v}_1 = (1, 0, 0)^T$, $\mathbf{v}_2 = (0, 1, 0)^T$, and $\mathbf{v}_3 = (1, 2, 0)^T$. Notice that

$$\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2.$$

- ♦ Therefore $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

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Linear Independence

- Three vectors are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
 - ♦ Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

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Linear Independence

Definition: The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k &= \mathbf{0} \\ \Rightarrow c_1 = c_2 = \cdots = c_k &= 0. \end{aligned}$$

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[Two vectors](#)

[Three vectors](#)

[More vectors](#)

When are $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k Linearly Independent?

- Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k as its columns.
- Find the nullspace, $\text{null}(M)$.
 - If $\text{null}(M) = \{\mathbf{0}\}$, the vectors are linearly independent.
 - If $\mathbf{a} \in \text{null}(M)$, and $\mathbf{a} = (a_1, a_2, \dots, a_k)^T \neq \mathbf{0}$, then

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_k \mathbf{v}_k = \mathbf{0}$$

and the vectors are linearly dependent.

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Definition

Matrix product

Example 1

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}$$

$$M = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} 1 & -1 & 5 \\ -2 & -3 & 0 \\ 0 & 2 & -4 \\ 2 & 0 & 6 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}(M)$ consists of all multiples of $\mathbf{a} = (-3, 2, 1)^T$.
- $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, so the vectors are linearly dependent.

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Method

Example 2

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

$$M = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} 1 & -1 & 5 \\ -2 & -3 & 0 \\ 0 & 2 & -4 \\ 2 & 0 & 3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = \{\mathbf{0}\}$.
- $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

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Method

Basis of a Subspace

Definition: A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k form a *basis* of a subspace V if

1. $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$
2. $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k are linearly independent.

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Examples of Bases

- The vector $\mathbf{v} = (1, -1, 1)^T$ is a basis for $\text{null}(A)$.
 - ♦ $\text{null}(A)$ is the subspace of \mathbf{R}^3 with basis \mathbf{v} .
- The vectors $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$ form a basis for $\text{null}(B)$.
 - ♦ $\text{null}(B)$ is the subspace of \mathbf{R}^4 with basis $\{\mathbf{v}, \mathbf{w}\}$.

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[Other examples](#)

Existence of a Basis

Proposition: Let V be a subspace of \mathbf{R}^n .

1. If $V \neq \{\mathbf{0}\}$, then V has a basis.
2. Bases are not unique, but every basis of V has the same number of elements.

Definition: The *dimension* of a subspace V is the number of elements in a basis of V .

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[Examples](#)

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Another Example of a Nullspace

$$A = \begin{pmatrix} 3 & -3 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 13 & -13 & 5 & -5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}(A)$ is the subspace of \mathbf{R}^4 with basis $(1, 1, 0, 0)^T$ and $(0, 0, 1, -1)^T$.
- $\text{null}(A)$ has dimension 2.
- In MATLAB, use commands `null(A)` or `null(A, 'r')`.

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Product of a Matrix with a Vector

- The *product* of a matrix A and a vector \mathbf{x} is the linear combination of the columns of A with the elements of \mathbf{x} as coefficients.
- Example:

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 13 \\ -5 \\ 23 \end{pmatrix} \\ = 13 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (-5) \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 23 \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

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