Math 211

Lecture #20

Nullspaces and Subspaces

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The Solution Set of $A\mathbf{x} = \mathbf{b}$

Theorem: Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector $\mathbf{x}_h$ such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

- Thus, the solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution $\mathbf{x}_p$ and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.

Return

The Nullspace of a Matrix

- The solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the *nullspace* of the matrix A. It is denoted by $\mathrm{null}(A)$. Thus

$$\mathrm{null}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}.$$

- What are the properties of nullspaces?
- Is there a convenient way to describe them?

Properties of Nullspaces

Proposition: Let A be a matrix.

1. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathrm{null}(A)$, then $\mathbf{x} + \mathbf{y}$ is in $\mathrm{null}(A)$.
2. If a is a scalar and $\mathbf{x}$ is in $\mathrm{null}(A)$, then $a\mathbf{x}$ is in $\mathrm{null}(A)$.

Definition: A nonempty subset V of $\mathbf{R}^n$ that has the properties

1. if $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, then $\mathbf{x} + \mathbf{y}$ is in V,
2. if a is a scalar, and $\mathbf{x}$ is in V, then $a\mathbf{x}$ is in V,

is called a *subspace* of $\mathbf{R}^n$.

Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through $\mathbf{0}$, $V = \{t\mathbf{v} \,|\, t \in \mathbf{R}\}$, is a subspace.
- A plane through $\mathbf{0}$, $V = \{a\mathbf{v} + b\mathbf{w} \,|\, a, b \in \mathbf{R}\}$, is a subspace.
- $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are subspaces of $\mathbf{R}^n$.
    - These are called the *trivial subspaces.*

Linear Combinations

Proposition: Any linear combination of vectors in a subspace V is also in V.

- Subspaces of $\mathbf{R}^n$ have the same linear structure as $\mathbf{R}^n$ itself.
- The nullspace of a matrix is a subspace, so it has the same linear structure as $\mathbf{R}^n$.
- The product of a matrix A and a vector $\mathbf{x}$ is the linear combination of the column vectors in A with the elements of $\mathbf{x}$ as coefficients.

Another Example of a Nullspace

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nullspace of A is the set

$$\text{null}(A) = \{a\mathbf{v} \,|\, a \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1)^T$.

- The nullspace of A consists of all multiples of $\mathbf{v}$.

Another Example of a Nullspace

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nullspace of B is the set

$$\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \,|\, a,\, b \in \mathbf{R}\}\,,$$

where $\mathbf{v} = (1,\, -1,\, 1,\, 0)^T$ and $\mathbf{w} = (0,\, -2,\, 0,\, 1)^T$.

- $\text{null}(B)$ consists of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$.

The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ is denoted by

$$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

Proposition: If $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are all vectors in $\mathbf{R}^n$, then $V = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbf{R}^n$.

How Do We Know if $\mathbf{w} \in \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ as its columns.
2. Solve the system $M\mathbf{a} = \mathbf{w}$.
   a. If there are no solutions, $\mathbf{w}$ is *NOT* in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.
   b. If there is a solution $\mathbf{a} = (a_1, a_2, \cdots, a_k)^T$, then
   $$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k$$
   is in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.

Examples of Spans

Let $\mathbf{v}_1 = (1,\ 2)^T$, $\mathbf{v}_2 = (1,\ 0)^T$, and $\mathbf{v}_3 = (2,\ 0)^T$.

- $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{R}^2$. (Proof?)
- $\text{span}(\mathbf{v}_1, \mathbf{v}_3) = \mathbf{R}^2$. (Proof?)
- $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{v}_2)$. (Proof?)
  - $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \{t\mathbf{v}_2 \,|\, t \in \mathbf{R}\}$.
  - $\mathbf{v}_2$ and $\mathbf{v}_3$ have the same direction.

Linear Independence of Two Vectors

We need a condition that will keep unneeded vectors out of a spanning list. We will work toward a general definition.

- Two vectors are *linearly dependent* if one is a scalar multiple of the other.
  - $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are *linearly independent.*

Linear Independence of Three Vectors

- Three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are *linearly dependent* if one is a linear combination of the other two.
  - Example: $\mathbf{v}_1 = (1,\, 0,\, 0)^T$, $\mathbf{v}_2 = (0,\, 1,\, 0)^T$, and $\mathbf{v}_3 = (1,\, 2,\, 0)^T$. Notice that

$$\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2.$$

  - Therefore $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

Linear Independence

- Three vectors are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
  - Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

Linear Independence

Definition: The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$
$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

When are $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ Linearly Independent?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ as its columns.
2. Find the nullspace, $\mathrm{null}(M)$.
   a. If $\mathrm{null}(M) = \{\mathbf{0}\}$, the vectors are linearly independent.
   b. If $\mathbf{a} \in \mathrm{null}(M)$, and $\mathbf{a} = (a_1, a_2, \cdots, a_k)^T \neq \mathbf{0}$, then

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k = \mathbf{0}$$

   and the vectors are linearly dependent.

Example 1

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}$$

$$M = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} 1 & -1 & 5 \\ -2 & -3 & 0 \\ 0 & 2 & -4 \\ 2 & 0 & 6 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\mathrm{null}(M)$ consists of all multiples of $\mathbf{a} = (-3, 2, 1)^T$.
- $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, so the vectors are linearly dependent.

Example 2

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

$$M = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} 1 & -1 & 5 \\ -2 & -3 & 0 \\ 0 & 2 & -4 \\ 2 & 0 & 3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = \{\mathbf{0}\}$.
- $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

Basis of a Subspace

Definition: A set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ form a *basis* of a subspace V if

1. $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$
2. $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are linearly independent.

Examples of Bases

- The vector $\mathbf{v} = (1,\ -1,\ 1)^T$ is a basis for $\mathrm{null}(A)$.
  - $\mathrm{null}(A)$ is the subspace of $\mathbf{R}^3$ with basis $\mathbf{v}$.
- The vectors $\mathbf{v} = (1,\ -1,\ 1,\ 0)^T$ and $\mathbf{w} = (0,\ -2,\ 0,\ 1)^T$ form a basis for $\mathrm{null}(B)$.
  - $\mathrm{null}(B)$ is the subspace of $\mathbf{R}^4$ with basis $\{\mathbf{v}, \mathbf{w}\}$.

Existence of a Basis

Proposition: Let V be a subspace of $\mathbf{R}^n$.

1. If $V \neq \{\mathbf{0}\}$, then V has a basis.
2. Bases are not unique, but every basis of V has the same number of elements.

Definition: The *dimension* of a subspace V is the number of elements in a basis of V.

Another Example of a Nullspace

$$A = \begin{pmatrix} 3 & -3 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 13 & -13 & 5 & -5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}(A)$ is the subspace of $\mathbf{R}^4$ with basis $(1, 1, 0, 0)^T$ and $(0, 0, 1, -1)^T$.
- $\text{null}(A)$ has dimension 2.
- In MATLAB, use commands `null(A)` or `null(A,'r')`.

Product of a Matrix with a Vector

- The *product* of a matrix A and a vector $\mathbf{x}$ is the linear combination of the columns of A with the elements of $\mathbf{x}$ as coefficients.

- Example:

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 13 \\ -5 \\ 23 \end{pmatrix} = 13 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (-5) \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 23 \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$