

Math 211

Lecture #21

Determinants

October 16, 2002

Basis of a Subspace

Definition: A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots,$ and \mathbf{v}_k form a *basis* of a subspace V if

1. $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$
 2. $\mathbf{v}_1, \mathbf{v}_2, \dots,$ and \mathbf{v}_k are linearly independent.
- The best way to describe a subspace is to give a basis.

[Return](#)

[Span](#)

Examples of Bases

- The vector $\mathbf{v} = (1, -1, 1)^T$ is a basis for $\text{null}(A)$.
 - ♦ $\text{null}(A)$ is the subspace of \mathbf{R}^3 with basis \mathbf{v} .
- The vectors $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$ form a basis for $\text{null}(B)$.
 - ♦ $\text{null}(B)$ is the subspace of \mathbf{R}^4 with basis $\{\mathbf{v}, \mathbf{w}\}$.

[Return](#)

Existence of a Basis

Proposition: Let V be a subspace of \mathbf{R}^n .

1. If $V \neq \{\mathbf{0}\}$, then V has a basis.
2. Bases are not unique, but every basis of V has the same number of elements.

Definition: The *dimension* of a subspace V is the number of elements in a basis of V .

Return

Examples

Another Example of a Nullspace

$$A = \begin{pmatrix} 3 & -3 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 13 & -13 & 5 & -5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}(A)$ is the subspace of \mathbf{R}^4 with basis $(1, 1, 0, 0)^T$ and $(0, 0, 1, 1)^T$.
- $\text{null}(A)$ has dimension 2.
- In MATLAB, use commands `null(A)` or `null(A, 'r')`.

Return

Nonsingular Matrices

Let A be an $n \times n$ matrix. We know the following:

- A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side \mathbf{b} . (This is the definition.)
- If A is nonsingular then $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side \mathbf{b} .
- A is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.
 - $\text{null}(A)$ is non-trivial $\Leftrightarrow A$ is singular.

Return

Theorem

Determinants in 2D

- How do we decide if a matrix A is nonsingular?
- A is nonsingular if and only if when put into row echelon form, the matrix has nonzero entries along the diagonal.
- Example: the general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is nonsingular if and only if $ad - bc \neq 0$.

- We define $ad - bc$ to be the *determinant* of A .

[Return](#)

Determinants in 3D

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- The same (but more difficult) argument shows that A is nonsingular if and only if

$$\begin{aligned} & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ & + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} \\ & \neq 0. \end{aligned}$$

- This will be the determinant of A .

[Return](#)

[2D](#)

Main Theorem

We will define the determinant of a square matrix A so that the next theorem is true.

Theorem: The $n \times n$ matrix A is nonsingular if and only if $\det(A) \neq 0$.

Corollary: If A is an $n \times n$ matrix, then $\text{null}(A)$ contains a nonzero vector if and only if $\det(A) = 0$.

- The corollary contains the most important fact about determinants for ODEs.

[Return](#)

Matrices and Minors

The general $n \times n$ matrix has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Definition: The ij -minor of an $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix A_{ij} obtained from A by deleting the i th row and the j th column.

Return

Definition of Determinant

Definition: The *determinant* of an $n \times n$ matrix A is defined to be

$$\det(A) = \sum_{j=1}^n (-1)^{j+1} a_{1j} \det(A_{1j}).$$

- The definition is inductive.
 - ♦ It assumes we know how to compute the determinants of $(n-1) \times (n-1)$ matrices.
 - ♦ We start with the 2×2 matrix.

Return

Matrix

3×3

Example

$$\begin{aligned} \det \begin{pmatrix} 2 & 1 & 0 \\ 3 & -2 & 4 \\ -1 & 5 & 3 \end{pmatrix} &= (-1)^2 \times 2 \times \det \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix} \\ &\quad + (-1)^3 \times 1 \times \det \begin{pmatrix} 3 & 4 \\ -1 & 3 \end{pmatrix} \\ &= 2 \times (-26) - 13 \\ &= -65 \end{aligned}$$

Definition

Expansion by the i^{th} Row

For any i , we have

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the i^{th} row*.
- Example:

$$\det \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & 0 \\ 2 & -16 & 9 \end{pmatrix} = 4 \cdot \det \begin{pmatrix} 5 & 3 \\ 2 & 9 \end{pmatrix} = 156.$$

[Return](#)

[Definition](#)

Properties of the Determinant

- The formula for the determinant of a matrix A is the sum of $n!$ products of the entries of A (sometimes $\times -1$.)
 - ♦ Each summand is the product of n entries, one from each row, and one from each column.
- The determinant of a triangular matrix is the product of the diagonal terms.
 - ♦ We can use row operations to compute determinants.

[Return](#)

Row Operations and Determinants

If B is obtained from A by

- adding a multiple of one row to another, then

$$\det(B) = \det(A).$$

- interchanging two rows, then

$$\det(B) = -\det(A).$$

- multiplying a row by $c \neq 0$, then

$$\det(B) = c \det(A).$$

[Return](#)

Example

$$A = \begin{pmatrix} -5 & 2 & 3 \\ 25 & -9 & -12 \\ 10 & 7 & 17 \end{pmatrix}$$

$$\det(A) = 50$$

Return

Row operations

More Properties

- If A has two equal rows, then $\det(A) = 0$.
- If A has a row of all zeros, then $\det(A) = 0$.
- $\det(A^T) = \det(A)$.
- If A has two equal columns, then $\det(A) = 0$.
- If A has a column of all zeros, then $\det(A) = 0$.

Return

Column Operations and Determinants

If B is obtained from A by

- adding a multiple of one column to another, then

$$\det(B) = \det(A).$$

- interchanging two columns, then

$$\det(B) = -\det(A).$$

- multiplying a column by $c \neq 0$, then

$$\det(B) = c \det(A).$$

Return

Expansion by a Column

We can also expand by a column.

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the j^{th} column*.

Return

 $\det(A^T) = \det(A)$

Expansion by row

Example

$$A = \begin{pmatrix} -5 & -6 & 0 \\ 3 & 4 & 0 \\ -8 & -16 & 9 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= 9 \cdot \det \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \\ &= 9 \cdot (-2) \\ &= -18 \end{aligned}$$

Return

Expansion by row

Expansion by column

Determinants and Bases

Proposition: A collection of n vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in \mathbf{R}^n is a basis for \mathbf{R}^n if and only if

$$\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n]) \neq 0.$$

Theorem

Examples

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix} = 1.$$

$$\det \begin{pmatrix} 3 & -1 & 0 & 1 \\ 12 & -6 & 0 & 5 \\ 32 & -15 & -3 & 13 \\ 18 & -10 & -1 & 8 \end{pmatrix} = -1.$$

Expansion by row Expansion by column Row operations Column operations

The Span of a Set of Vectors

Definition: The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).$$

Return

Linear Independence

Definition: The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$\begin{aligned} c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k &= \mathbf{0} \\ \Rightarrow c_1 = c_2 = \cdots = c_k &= 0. \end{aligned}$$

Return

Example of a Nullspace

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nullspace of A is the set

$$\text{null}(A) = \{a\mathbf{v} \mid a \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1)^T$.

- The nullspace of A consists of all multiples of \mathbf{v} .

Return

Another Example of a Nullspace

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nullspace of B is the set

$$\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$.

- $\text{null}(B)$ consists of all linear combinations of \mathbf{v} and \mathbf{w} .

Return