# Math 211

Lecture #21

Determinants

October 16, 2002

# Basis of a Subspace

**Definition:** A set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ form a *basis* of a subspace $V$ if

1. $V = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$
2. $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are linearly independent.

- The best way to describe a subspace is to give a basis.

# Examples of Bases

- The vector $\mathbf{v} = (1,\ -1,\ 1)^T$ is a basis for $\text{null}(A)$.
  - $\text{null}(A)$ is the subspace of $\mathbf{R}^3$ with basis $\mathbf{v}$.
- The vectors $\mathbf{v} = (1,\ -1,\ 1,\ 0)^T$ and $\mathbf{w} = (0,\ -2,\ 0,\ 1)^T$ form a basis for $\text{null}(B)$.
  - $\text{null}(B)$ is the subspace of $\mathbf{R}^4$ with basis $\{\mathbf{v}, \mathbf{w}\}$.

# Existence of a Basis

**Proposition:** Let $V$ be a subspace of $\mathbf{R}^n$.

1. If $V \neq \{\mathbf{0}\}$, then $V$ has a basis.
2. Bases are not unique, but every basis of $V$ has the same number of elements.

**Definition:** The *dimension* of a subspace $V$ is the number of elements in a basis of $V$.

# Another Example of a Nullspace

$$A = \begin{pmatrix} 3 & -3 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 13 & -13 & 5 & -5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}(A)$ is the subspace of $\mathbf{R}^4$ with basis $(1,1,0,0)^T$ and $(0,0,1,1)^T$.
- $\text{null}(A)$ has dimension 2.
- In MATLAB, use commands `null(A)` or `null(A,'r')`.

# Nonsingular Matrices

Let $A$ be an $n \times n$ matrix. We know the following:

- $A$ is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side $\mathbf{b}$. (This is the definition.)
- If $A$ is nonsingular then $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side $\mathbf{b}$.
- $A$ is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.
  - $\text{null}(A)$ is non-trivial $\Leftrightarrow$ $A$ is singular.

# Determinants in 2D

- How do we decide if a matrix $A$ is nonsingular?
- $A$ is nonsingular if and only if when put into row echelon form, the matrix has nonzero entries along the diagonal.
- Example: the general $2 \times 2$ matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is nonsingular if and only if $ad - bc \neq 0$.

  - We define $ad - bc$ to be the *determinant* of $A$.

# Determinants in 3D

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- The same (but more difficult) argument shows that $A$ is nonsingular if and only if

$$\begin{aligned} & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ & \quad + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} \\ & \neq 0. \end{aligned}$$

- This will be the determinant of $A$.

# Main Theorem

We will define the determinant of a square matrix $A$ so that the next theorem is true.

**Theorem:** The $n \times n$ matrix $A$ is nonsingular if and only if $\det(A) \neq 0$.

**Corollary:** If $A$ is an $n \times n$ matrix, then $\text{null}(A)$ contains a nonzero vector if and only if $\det(A) = 0$.

- The corollary contains the most important fact about determinants for ODEs.

# Matrices and Minors

The general $n \times n$ matrix has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

**Definition:** The *$ij$-minor* of an $n \times n$ matrix $A$ is the $(n-1) \times (n-1)$ matrix $A_{ij}$ obtained from $A$ by deleting the $i^{\text{th}}$ row and the $j^{\text{th}}$ column.

# Definition of Determinant

**Definition:** The *determinant* of an $n \times n$ matrix $A$ is defined to be

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_{1j}).$$

- The definition is inductive.
  - It assumes we know how to compute the determinants of $(n-1) \times (n-1)$ matrices.
  - We start with the $2 \times 2$ matrix.

# Example

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 3 & -2 & 4 \\ -1 & 5 & 3 \end{pmatrix}$$

$$= (-1)^2 \times 2 \times \det\begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix}$$

$$+ (-1)^3 \times 1 \times \det\begin{pmatrix} 3 & 4 \\ -1 & 3 \end{pmatrix}$$

$$= 2 \times (-26) - 13$$

$$= -65$$

# Expansion by the $i^{\text{th}}$ Row

For any $i$, we have

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the $i^{th}$ row.*
- Example:

$$\det \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & 0 \\ 2 & -16 & 9 \end{pmatrix} = 4 \cdot \det \begin{pmatrix} 5 & 3 \\ 2 & 9 \end{pmatrix} = 156.$$

# Properties of the Determinant

- The formula for the determinant of a matrix $A$ is the sum of $n!$ products of the entries of $A$ (sometimes $\times -1.$)
  - Each summand is the product of $n$ entries, one from each row, and one from each column.
- The determinant of a triangular matrix is the product of the diagonal terms.
  - We can use row operations to compute determinants.

# Row Operations and Determinants

If $B$ is obtained from $A$ by

- adding a multiple of one row to another, then

$$\det(B) = \det(A).$$

- interchanging two rows, then

$$\det(B) = -\det(A).$$

- multiplying a row by $c \neq 0$, then

$$\det(B) = c\det(A).$$

# Example

$$A = \begin{pmatrix} -5 & 2 & 3 \\ 25 & -9 & -12 \\ 10 & 7 & 17 \end{pmatrix}$$

$$\det(A) = 50$$

# More Properties

- If $A$ has two equal rows, then $\det(A) = 0$.
- If $A$ has a row of all zeros, then $\det(A) = 0$.
- $\det(A^T) = \det(A)$.
- If $A$ has two equal columns, then $\det(A) = 0$.
- If $A$ has a column of all zeros, then $\det(A) = 0$.

# Column Operations and Determinants

If $B$ is obtained from $A$ by

- adding a multiple of one column to another, then

$$\det(B) = \det(A).$$

- interchanging two columns, then

$$\det(B) = -\det(A).$$

- multiplying a column by $c \neq 0$, then

$$\det(B) = c\det(A).$$

# Expansion by a Column

We can also expand by a column.

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the $j^{th}$ column.*

# Example

$$A = \begin{pmatrix} -5 & -6 & 0 \\ 3 & 4 & 0 \\ -8 & -16 & 9 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= 9 \cdot \det \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \\ &= 9 \cdot (-2) \\ &= -18 \end{aligned}$$

# Determinants and Bases

**Proposition:** A collection of $n$ vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . ,$\mathbf{v}_n$ in $\mathbf{R}^n$ is a basis for $\mathbf{R}^n$ if and only if

$$\det([\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n]) \neq 0.$$

# Examples

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix} = 1.$$

$$\det\begin{pmatrix} 3 & -1 & 0 & 1 \\ 12 & -6 & 0 & 5 \\ 32 & -15 & -3 & 13 \\ 18 & -10 & -1 & 8 \end{pmatrix} = -1.$$

# The Span of a Set of Vectors

**Definition:** The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

# Linear Independence

**Definition:** The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$
$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

# Example of a Nullspace

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nullspace of $A$ is the set

$$\text{null}(A) = \{a\mathbf{v} \,|\, a \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1)^T$.

- The nullspace of $A$ consists of all multiples of $\mathbf{v}$.

# Another Example of a Nullspace

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nullspace of $B$ is the set

$$\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \,|\, a,\, b \in \mathbf{R}\}\,,$$

where $\mathbf{v} = (1,\, -1,\, 1,\, 0)^T$ and $\mathbf{w} = (0,\, -2,\, 0,\, 1)^T$.

- $\text{null}(B)$ consists of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$.