Math 211

Lecture #22

Systems of ODEs

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Systems of Differential Equations

Example: A mixed population of predators (foxes) and prey (rabbits).

- The prey, $x(t)$, flourish in the absence of the predators.
- The predators, $y(t)$, depend on the prey as a food source, and would die out in the absence of the prey.
- For predation to take place there must be an encounter between a predator and a prey.

Return

Predator-Prey Model

The basic model is $x' = r_x \cdot x$ and $y' = r_y \cdot y$, where r_x and r_y are the reproductive rates.

- $r_x = a > 0$ if $y = 0$, and decreases as y increases.
  - $r_x = a - by$.
- $r_y = -c < 0$ if $x = 0$, and increases as x increases.
  - $r_y = -c + dx$.
- The system becomes: $$x' = (a - by)x$$ $$y' = (-c + dx)y$$
- MATLAB & `pplane6`.

Return Assumptions

General System in 2D

$$x' = f(t, x, y)$$
$$y' = g(t, x, y)$$

- Example:

$$x' = y$$
$$y' = -x$$

- Solution: $x(t) = \sin t$ and $y(t) = \cos t$
  - Verify by direct substitution.

Return

General System in Higher D

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots &= \quad \vdots \\ x_n' &= f_n(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

- The *dimension* of a system is the number of unknown functions = the number of equations.
  - The predator-prey model has dimension 2.

Return Planar system

Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$, and

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

- Then in the example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

Return

Vector Notation — Planar System

- For the general case use vector notation and set

$$\mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} f(t, u_1, u_2) \\ g(t, u_1, u_2) \end{pmatrix}.$$

- Then

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

Return

Vector Notation — General

- In higher dimensions, set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

Return

Vector Notation — Predator-Prey Model

For the predator-prey model set $u_1 = x$, and $u_2 = y$.
Then the system can be written

$$\mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} (a - bu_2)u_1 \\ (-c + bu_1)u_2 \end{pmatrix}.$$

- This is an *autonomous* system.
  - The RHS has no explicit dependence on t.

Return
General case

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Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example:

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \text{with} \quad \begin{aligned} x(0) &= 2 \\ y(0) &= 13 \end{aligned}$$

- PP model: Both the initial prey population and the initial predator population must be specified.

Return

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Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

Return

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Example of Reduction

- Third-order equation: $y''' + 2yy' = 3\cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 3\cos t - 2x_1x_2 \end{aligned}$$

- This system is *not* autonomous.

Return

Geometric Interpretation of Solutions

- Parametric plot
  - Tangent vectors
- Vector fields
- Phase plane
- `pplane6` for planar autonomous systems.