

Math 211

Lecture #22

Systems of ODEs

October 18, 2002

Systems of Differential Equations

Example: A mixed population of predators (foxes) and prey (rabbits).

- The prey, $x(t)$, flourish in the absence of the predators.
- The predators, $y(t)$, depend on the prey as a food source, and would die out in the absence of the prey.
- For predation to take place there must be an encounter between a predator and a prey.

Predator-Prey Model

The basic model is $x' = r_x \cdot x$ and $y' = r_y \cdot y$, where r_x and r_y are the reproductive rates.

- $r_x = a > 0$ if $y = 0$, and decreases as y increases.
 - ◆ $r_x = a - by$.
- $r_y = -c < 0$ if $x = 0$, and increases as x increases.
 - ◆ $r_y = -c + dx$.
- The system becomes: $x' = (a - by)x$
 $y' = (-c + dx)y$
- MATLAB & ppplane6.

General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example:

$$x' = y$$

$$y' = -x$$

- Solution: $x(t) = \sin t$ and $y(t) = \cos t$
 - ◆ Verify by direct substitution.

General System in Higher D

$$x'_1 = f_1(t, x_1, x_2, \dots, x_n)$$

$$x'_2 = f_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots = \quad \quad \quad \vdots$$

$$x'_n = f_n(t, x_1, x_2, \dots, x_n)$$

- The *dimension* of a system is the number of unknown functions = the number of equations.
 - ♦ The *predator-prey model* has dimension 2.

Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$, and

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

- Then in the example

$$\begin{array}{l} x' = y \\ y' = -x \end{array} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

Vector Notation — Planar System

- For the **general case** use **vector** notation and set

$$\mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} f(t, u_1, u_2) \\ g(t, u_1, u_2) \end{pmatrix}.$$

- Then

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

Vector Notation — General

- In **higher** dimensions, set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix} .$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

Vector Notation — Predator-Prey Model

For the **predator-prey model** set $u_1 = x$, and $u_2 = y$.

Then the system can be written

$$\mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} (a - bu_2)u_1 \\ (-c + bu_1)u_2 \end{pmatrix}.$$

- This is an **autonomous** system.
 - ♦ The RHS has no explicit dependence on t .

Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each **component** of $\mathbf{x}(t_0)$ must be specified.
- **Example:**

$$\begin{array}{l} x' = y \\ y' = -x \end{array} \quad \text{with} \quad \begin{array}{l} x(0) = 2 \\ y(0) = 13 \end{array}$$

- **PP model:** Both the initial prey population and the initial predator population must be specified.

Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

Example of Reduction

- Third-order equation: $y''' + 2yy' = 3 \cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3 \cos t - 2x_1x_2$$

- This system is *not* autonomous.

Geometric Interpretation of Solutions

- Parametric plot
 - ◆ Tangent vectors
- Vector fields
- Phase plane
- `pplane6` for planar autonomous systems.