

# Math 211

Lecture #23  
Qualitative Analysis

October 21, 2002

## General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example:

$$x' = y$$

$$y' = -x$$

Return

## General System in Higher D

$$x'_1 = f_1(t, x_1, x_2, \dots, x_n)$$

$$x'_2 = f_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots = \quad \vdots$$

$$x'_n = f_n(t, x_1, x_2, \dots, x_n)$$

Return

Planar system

## Vector Notation — General

- Set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

Return

## Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of  $\mathbf{x}(t_0)$  must be specified.
- Example

$$\begin{array}{l} x' = y \\ y' = -x \end{array} \quad \text{with} \quad \begin{array}{l} x(0) = 2 \\ y(0) = 13 \end{array}$$

Return

## Geometric Interpretation of Solutions

- Parametric plot
  - ♦ Tangent vectors
- Vector fields
- Phase plane
- plane for planar autonomous systems.

## Existence & Uniqueness

General System  $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$

- $\mathbf{x}$  in an open set  $U \subset \mathbf{R}^n$
- $t$  in an interval  $I = (a, b)$

$$R = I \times U = \{(t, \mathbf{x}) \mid t \in I \text{ and } \mathbf{x} \in U\}.$$

[Return](#)

**Theorem:** Suppose that  $\mathbf{f}(t, \mathbf{x})$  is continuous in  $R$ , and that all first partials of  $\mathbf{f}$  are also continuous in  $R$ . Then given any  $t_0 \in I$  and  $\mathbf{x}_0 \in U$  there is a *unique* solution to the initial value problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

defined on an interval containing  $t_0$ . The solution exists at least until the solution curve  $t \rightarrow (t, \mathbf{x}(t))$  leaves  $R$ .

[Return](#)

[Previous](#)

## Autonomous Systems

System of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- Look at solution curves  $t \rightarrow \mathbf{x}(t) \in \mathbf{R}^n$ .
- $\mathbf{R}^n$  is called *phase space*.
  - ♦ If  $n = 2$ ,  $\mathbf{R}^2$  is the *phase plane*.
  - ♦ If  $n = 1$ ,  $\mathbf{R}^1$  is the *phase line*.

[Return](#)

## Uniqueness in Phase Space

Two solution curves in phase space for an autonomous system cannot meet at a point unless the solution curves coincide.

- If  $n = 2$ , two solution curves in the phase plane cannot cross, or even touch.
- *If the system is not autonomous, solution curves in the phase plane can cross.*

Return

E &amp; U

## Equilibrium Points & Solutions

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- The system is autonomous.
- $\mathbf{x}_0$  is an *equilibrium point* if  $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ .
- $\mathbf{x}(t) = \mathbf{x}_0$  is the corresponding *equilibrium solution*.
- In phase space, an equilibrium solution plots as a point.

Return

## Nullclines

A *nullcline* is the set where one component of the right-hand side of  $\mathbf{x}' = \mathbf{f}(\mathbf{x})$  vanishes.

- Example

$$x' = x^2 - y$$

$$y' = x - xy$$

- $x$ -nullcline:  $x^2 - y = 0$ .
- $y$ -nullcline:  $x(1 - y) = 0$ .
- 3 equilibrium points:  $(0, 0)$ ,  $(1, 1)$ , and  $(-1, 1)$ .

Return

## Linear Systems

A system is *linear* if the unknown functions appear linearly in the right-hand sides.

- *Appear linearly* means that there are no products, powers, or higher order functions.
- Examples
  - ♦ The Lotka-Volterra system is nonlinear.
  - ♦ Previous example is linear.

[Return](#)

## Planar Linear Systems

A planar linear system is one of the form

$$x' = a(t)x + b(t)y + f(t)$$

$$y' = c(t)x + d(t)y + g(t)$$

- The coefficients can depend on  $t$ .

[Return](#)

## General Linear Systems

$$x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1$$

$$x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2$$

$$\vdots = \quad \vdots$$

$$x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n$$

- The coefficients can depend on  $t$ .

[Return](#)

[General system](#)

[Linear system](#)

[2D linear system](#)

- Set

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \dots, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes  $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ .

[Return](#)

[Previous](#)

[General system](#)

## Existence & Uniqueness

**Theorem:** Suppose the matrix-valued function  $A = A(t)$  and the vector-valued function  $\mathbf{f}(t)$  are defined and continuous in an interval  $I = (\alpha, \beta)$ . Then for any  $t_0$  in  $I$  and any  $\mathbf{x}_0$  in  $\mathbf{R}^n$ , the initial value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution defined for all  $t$  in  $I$ .

[Return](#)

[Existence & Uniqueness](#)

## Homogeneous Systems

A *homogeneous* system is one of the form

$$\mathbf{x}' = A\mathbf{x}$$

**Proposition:** Suppose that  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ , and  $\mathbf{x}_k(t)$  are solutions to the homogeneous system, and  $c_1, c_2, \dots$ , and  $c_k$  are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

[Return](#)