Math 211

Lecture #23

Qualitative Analysis

October 21, 2002

General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example:

$$x' = y$$

$$y' = -x$$

General System in Higher D

$$
\begin{aligned}
x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\
x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\
\vdots &= \qquad \vdots \\
x_n' &= f_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}
$$

Vector Notation — General

- Set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{f}(t,\mathbf{x}) = \begin{pmatrix} f_1(t,\mathbf{x}) \\ f_2(t,\mathbf{x}) \\ \vdots \\ f_n(t,\mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t,\mathbf{x}).$$

Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \text{with} \quad \begin{aligned} x(0) &= 2 \\ y(0) &= 13 \end{aligned}$$

Geometric Interpretation of Solutions

- Parametric plot
  - Tangent vectors
- Vector fields
- Phase plane
- `pplane6` for planar autonomous systems.

Existence & Uniqueness

General System $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$

- $\mathbf{x}$ in an open set $U \subset \mathbf{R}^n$
- t in an interval $I = (a, b)$

$$R = I \times U = \{(t, x) \,|\, t \in I \text{ and } \mathbf{x} \in U\}.$$

Theorem: Suppose that $\mathbf{f}(t, \mathbf{x})$ is continuous in R, and that all first partials of $\mathbf{f}$ are also continuous in R. Then given any $t_0 \in I$ and $\mathbf{x}_0 \in U$ there is a *unique* solution to the initial value problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

defined on an interval containing t_0. The solution exists at least until the solution curve $t \to (t, \mathbf{x}(t))$ leaves R.

Autonomous Systems

System of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- Look at solution curves $t \longrightarrow \mathbf{x}(t) \in \mathbf{R}^n$.
- $\mathbf{R}^n$ is called *phase space.*
  - If $n = 2$, $\mathbf{R}^2$ is the *phase plane.*
  - If $n = 1$, $\mathbf{R}^1$ is the phase line.

Uniqueness in Phase Space

Two solution curves in phase space for an autonomous system cannot meet at a point unless the solution curves coincide.

- If $n = 2$, two solution curves in the phase plane cannot cross, or even touch.
- *If the system is not autonomous, solution curves in the phase plane can cross.*

Equilibrium Points & Solutions

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- The system is autonomous.
- $\mathbf{x}_0$ is an *equilibrium point* if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.
- $\mathbf{x}(t) = \mathbf{x}_0$ is the corresponding *equilibrium solution*.
- In phase space, an equilibrium solution plots as a point.

Nullclines

A *nullcline* is the set where one component of the right-hand side of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ vanishes.

- Example

$$x' = x^2 - y$$

$$y' = x - xy$$

- x-nullcline: $x^2 - y = 0$.
- y-nullcline: $x(1 - y) = 0$.
- 3 equilibrium points: $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

Linear Systems

A system is *linear* if the unknown functions appear linearly in the right-hand sides.

- *Appear linearly* means that there are no products, powers, or higher order functions.
- Examples
  - The Lotka-Volterra system is nonlinear.
  - Previous example is linear.

Planar Linear Systems

A planar linear system is one of the form

$$x' = a(t)x + b(t)y + f(t)$$
$$y' = c(t)x + d(t)y + g(t)$$

- The coefficients can depend on t.

General Linear Systems

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \\
\vdots &= \qquad \vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n
\end{aligned}
$$

- The coefficients can depend on t.

- Set

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Existence & Uniqueness

Theorem: Suppose the matrix-valued function $A = A(t)$ and the vector-valued function $\mathbf{f}(t)$ are defined and continuous in an interval $I = (\alpha, \beta)$. Then for any t_0 in I and any $\mathbf{x}_0$ in $\mathbf{R}^n$, the initial value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution defined *for all t in I.*

Homogeneous Systems

A *homogeneous* system is one of the form

$$\mathbf{x}' = A\mathbf{x}$$

Proposition: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_k(t)$ are solutions to the homogeneous system, and c_1, c_2, . . . , and c_k are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.