Math 211

Lecture #24

Linear Systems of ODEs

October 23, 2002

General Linear Systems

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \\
\vdots &= \qquad \vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n
\end{aligned}
$$

- The coefficients can depend on t.

- Set

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Homogeneous Systems

An *homogeneous* system is one of the form

$$\mathbf{x}' = A\mathbf{x}$$

Proposition: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_k(t)$ are solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$, and c_1, c_2, . . . , and c_k are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

Very Important Example

- The system

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

has solutions

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

  - Verify by direct substitution.
- Proposition $\Rightarrow \mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ is a solution for any constants C_1 and C_2.
  - Is this the general solution?

- Let $\mathbf{y}$ be a solution of $\mathbf{y}' = A\mathbf{y}$. Can we find C_1 and C_2 so that $\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ for all t?
- Let's ask a simpler question. Can we find C_1 and C_2 so that $\mathbf{y}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)$?
  - Yes, since $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.
- Uniqueness theorem $\Rightarrow$

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t.$$

- Thus, every solution to $\mathbf{x}' = A\mathbf{x}$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.
- Can we generalize this result?

Key Point in the Argument

- Need to solve the equation

$$\mathbf{y}_0 = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)$$

for any $\mathbf{y}_0 = \mathbf{y}(0)$.

- Possible if $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.
- Uniqueness then implies that

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t \quad .$$

- We needed $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to be linearly independent at only one point.

Return

Proposition: $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_k(t)$ solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I.

1. If $\mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, . . . , and $\mathbf{x}_k(t_0)$ are linearly independent for some $t_0 \in I$, then they are linearly independent for all $t \in I$.

2. If $\mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, . . . , and $\mathbf{x}_k(t_0)$ are linearly dependent for some $t_0 \in I$, then they are linearly dependent for all $t \in I$.

Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is *linearly independent* if they are linearly independent at one value of t.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of t.

Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I. Then every solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$.

- That is, if $\mathbf{x}(t)$ is any solution, then there are constants C_1, C_2, . . . , and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t).$$

- The general solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$.

Solution Strategy

- The obvious strategy for completely solving an $n \times n$ homogeneous system is to look for n linearly independent solutions.

Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a *fundamental set of solutions.*

- We will develop methods of finding fundamental sets of solutions.

Examples: $\mathbf{x}' = A\mathbf{x}$

- Example 1: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

is a fundamental set of solutions.

- Example 2: $A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$

$$\mathbf{x}_1(t) = e^{t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is a fundamental set of solutions.

Linear Systems with Constant Coefficients

- We will solve homogeneous systems, $\mathbf{x}' = A\mathbf{x}$, first.
- We will be able to find explicit solutions.
- To motivate what we do, we will start with the easiest case, dimension $= 1$.
  - One equation: $x' = ax$, where a is a constant.
  - Solution: $x(t) = Ce^{at}$
  - All solutions are exponentials. Can we find exponential solutions to a system of equations?

Exponential Solutions to $\mathbf{x}' = A\mathbf{x}$

- Look for solution of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where $\mathbf{v}$ is a vector with constant entries.
- Substituting we get

$$\mathbf{x}' = \lambda e^{\lambda t}\mathbf{v}$$

$$A\mathbf{x} = e^{\lambda t}A\mathbf{v}$$

- Hence $\mathbf{x}' = A\mathbf{x} \quad \Leftrightarrow \quad A\mathbf{v} = \lambda\mathbf{v}$
- If $A\mathbf{v} = \lambda\mathbf{v}$ then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Can we find λ and $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$?

Eigenvalues & Eigenvectors

Definition: λ is an *eigenvalue* of A if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. If λ is an eigenvalue of A, then any vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with* λ.

- If λ an eigenvalue of A, and $\mathbf{v}$ is an associated nonzero eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.
  - Thus we have a way to find some solutions to systems with constant coefficients.
- How do we find eigenvalues and eigenvectors?

Finding Eigenvalues

λ is an eigenvalue of A

$\Leftrightarrow$ there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

$\Leftrightarrow$ $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{0} = A\mathbf{v} - \lambda\mathbf{v}$

$$= A\mathbf{v} - \lambda I\mathbf{v}$$

$$= (A - \lambda I)\mathbf{v}$$

$\Leftrightarrow$ $A - \lambda I$ has a nontrivial nullspace.

$\Leftrightarrow$ $\det(A - \lambda I) = 0$.

Example

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -4 - \lambda & 2 \\ -3 & 1 - \lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (-4 - \lambda)(1 - \lambda) + 6 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 1)(\lambda + 2) \end{aligned}$$

- A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

The Characteristic Polynomial

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix}$$

- If A is an $n \times n$ matrix $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n.

Definition: The *characteristic polynomial* of the $n \times n$ matrix A is

$$p(\lambda) = \det(A - \lambda I).$$

The *characteristic equation* is $p(\lambda) = 0$.

- Thus, the eigenvalues of A are the roots of the characteristic equation.

Our Solution Strategy for $\mathbf{x}' = A\mathbf{x}$

If A is $n \times n$, we are looking for n linearly independent solutions.

- Each eigenvalue λ of A has by definition an associated nonzero eigenvector $\mathbf{v}$. This gives us the solution , $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.
- The eigenvalues of A are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$.
  - ♦ $p(\lambda)$ has degree n, and usually has n roots.
- Therefore, there are usually n different solutions.
  - ♦ Are they linearly independent?

Finding Eigenvectors

$\mathbf{v}$ is an eigenvector associated with the eigenvalue λ if

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$\Leftrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$$

$$\Leftrightarrow \mathbf{v} \in \mathrm{null}(A - \lambda I)$$

- The set of all eigenvectors associated to the eigenvalue λ is equal to the nullspace of $A - \lambda I$.
  - It is a subspace of $\mathbf{R}^n$.
  - It is called the *eigenspace* of λ.

Example: A $= \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$

A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

- $\lambda_1 = -1$: $A - \lambda_1 I = \begin{pmatrix} -4+1 & 2 \\ -3 & 1+1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}$
  - $\mathbf{v}_1 = (2,3)^T$ is an eigenvector .
  - $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}(2,3)^T$ is a solution .
- $\lambda_2 = -2$: $A - \lambda_2 I = \begin{pmatrix} -4+2 & 2 \\ -3 & 1+2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$
  - $\mathbf{v}_2 = (1,1)^T$ is an eigenvector .
  - $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t}(1,1)^T$ is a solution .

Example (cont.)

- $x_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent .
- $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions.
- The general solution is the set of all linear combinations:

$$\begin{aligned} \mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \\ &= C_1 e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2C_1 e^{-t} + C_2 e^{-2t} \\ 3C_1 e^{-t} + C_2 e^{-2t} \end{pmatrix} \end{aligned}$$

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If λ is an eigenvalue and $\mathbf{v}$ is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, *if we can.*
    - This must be explored further.