Math 211

Lecture #25

Exponential Solutions

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Homogeneous Systems

- These are systems of the form

$$\mathbf{x}' = A\mathbf{x},$$

  where A is an $n \times n$ matrix.

- We are looking primarily at homogeneous systems with constant coefficients.

Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I. Then every solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$.

- That is, if $\mathbf{x}(t)$ is a solution, then there are constants C_1, C_2, . . . , and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t).$$

Solution Strategy

- The obvious strategy for completely solving the system is to look for n linearly independent solutions. solutions.

Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a *fundamental set of solutions.*

- We will look for fundamental sets of solutions.

Exponential Solutions to $\mathbf{x}' = A\mathbf{x}$

- Look for solution of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where $\mathbf{v}$ is a vector with constant entries.
- Substituting we get

$$\mathbf{x}' = \lambda e^{\lambda t}\mathbf{v}$$
$$A\mathbf{x} = e^{\lambda t}A\mathbf{v}$$

- Hence $\mathbf{x}' = A\mathbf{x} \quad \Leftrightarrow \quad A\mathbf{v} = \lambda\mathbf{v}$
- If $A\mathbf{v} = \lambda\mathbf{v}$ then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Can we find λ and $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$?

Eigenvalues & Eigenvectors

Definition: λ is an *eigenvalue* of A if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. If λ is an eigenvalue of A, then any vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with* λ.

- If λ an eigenvalue of A, and $\mathbf{v}$ is an associated nonzero eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.
  - Thus we have a way to find some solutions to systems with constant coefficients.
- How do we find eigenvalues and eigenvectors?

Finding Eigenvalues

λ is an eigenvalue of A

$\Leftrightarrow$ there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

$\Leftrightarrow$ $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{0} = A\mathbf{v} - \lambda\mathbf{v}$

$$= A\mathbf{v} - \lambda I\mathbf{v}$$

$$= (A - \lambda I)\mathbf{v}$$

$\Leftrightarrow$ $A - \lambda I$ has a nontrivial nullspace.

$\Leftrightarrow$ $\det(A - \lambda I) = 0$.

Example

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -4 - \lambda & 2 \\ -3 & 1 - \lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (-4 - \lambda)(1 - \lambda) + 6 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 1)(\lambda + 2) \end{aligned}$$

- A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

The Characteristic Polynomial

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix}$$

- If A is an $n \times n$ matrix $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n.

Definition: The *characteristic polynomial* of the $n \times n$ matrix A is

$$p(\lambda) = \det(A - \lambda I).$$

The *characteristic equation* is $p(\lambda) = 0$.

- Thus, the eigenvalues of A are the roots of the characteristic equation.

Our Solution Strategy for $\mathbf{x}' = A\mathbf{x}$

If A is $n \times n$, we are looking for n linearly independent solutions.

- Each eigenvalue λ of A has by definition an associated nonzero eigenvector $\mathbf{v}$. This gives us the solution , $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.
- The eigenvalues of A are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$.
  - $p(\lambda)$ has degree n, and usually has n roots.
- Therefore, there are usually n different solutions.
  - Are they linearly independent?

Finding Eigenvectors

$\mathbf{v}$ is an eigenvector associated with the eigenvalue λ if

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$\Leftrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$$

$$\Leftrightarrow \mathbf{v} \in \text{null}(A - \lambda I)$$

- The set of all eigenvectors associated to the eigenvalue λ is equal to the nullspace of $A - \lambda I$.
  - It is a subspace of $\mathbf{R}^n$.
  - It is called the *eigenspace* of λ.

Example: A $= \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$

A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

- $\lambda_1 = -1$: $A - \lambda_1 I = \begin{pmatrix} -4+1 & 2 \\ -3 & 1+1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}$
  - $\mathbf{v}_1 = (2,3)^T$ is an eigenvector .
  - $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}(2,3)^T$ is a solution .
- $\lambda_2 = -2$: $A - \lambda_2 I = \begin{pmatrix} -4+2 & 2 \\ -3 & 1+2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$
  - $\mathbf{v}_2 = (1,1)^T$ is an eigenvector .
  - $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t}(1,1)^T$ is a solution .

Example (cont.)

- $x_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent .
- $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions.
- The general solution is the set of all linear combinations:

$$\begin{aligned}
\mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \\
&= C_1 e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2C_1 e^{-t} + C_2 e^{-2t} \\ 3C_1 e^{-t} + C_2 e^{-2t} \end{pmatrix}
\end{aligned}$$

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If λ is an eigenvalue and $\mathbf{v}$ is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, *if we can.*
  - This must be explored further.

Solving $\mathbf{x}' = A\mathbf{x}$

Cases to be Considered

- Distinct real eigenvalues.
  - In this case the method works as described.
- Complex eigenvalues.
  - The method yields complex solutions, but we will want real solutions.
- Repeated eigenvalues.
  - The method does not always give enough solutions.
    - This is the hard case.

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

In nonvector form

$$x_1' = a_{11}x_1 + a_{12}x_2$$

$$x_2' = a_{21}x_1 + a_{22}x_2$$

The Characteristic Polynomial

$$
\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \\
&= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \\
&= \lambda^2 - T\lambda + D,
\end{aligned}
$$

where

- $D = a_{11}a_{22} - a_{12}a_{21} = \det(A)$
- $T = a_{11} + a_{22} = \mathrm{tr}(A)$ is the *trace* of A.
  - The *trace* of a matrix is the sum of its diagonal elements.

The Eigenvalues of A

- The eigenvalues of A are the roots of the characteristic equation $p(\lambda) = \lambda^2 - T\lambda + D = 0.$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Three cases:
  - 2 distinct real roots if $T^2 - 4D > 0$
  - 2 complex conjugate roots if $T^2 - 4D < 0$
  - Double real root if $T^2 - 4D = 0$

Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

Proposition: Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix A, and that $\mathbf{v}_1 \neq 0$ and $\mathbf{v}_2 \neq 0$ are eigenvectors associated with λ_1 and λ_2, respectively. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Two Distinct Real Eigenvalues

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- $T^2 - 4D > 0$ so $\lambda_1 < \lambda_2$.
- There are associated nonzero eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
- Solutions $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$.
- $\mathbf{x}_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent; $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ form a fundamental set of solutions.
- The general solution is $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$.

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix}$$

- Characteristic polynomial: $p(\lambda) = \lambda^2 - 4$.
- Eigenvalues: $\lambda_1 = -2$ and $\lambda_2 = 2$.
  - $\lambda_1 = -2$. Eigenvector: $\mathbf{v}_1 = (-2, 1)^T$.
    - Solution: $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-2t}(-2, 1)^T$.
  - $\lambda_2 = 2$. Eigenvector: $\mathbf{v}_2 = (-1, 1)^T$.
    - Solution: $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{2t}(-1, 1)^T$.

- $\mathbf{x}_1$ and $\mathbf{x}_2$ are a fundamental set of solutions.
- The general solution is

$$\begin{aligned}\mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)\\ &= C_1e^{-2t}\begin{pmatrix}-2\\ 1\end{pmatrix} + C_2e^{2t}\begin{pmatrix}-1\\ 1\end{pmatrix}.\end{aligned}$$

Initial Value Problem

Solve $\mathbf{x}' = A\mathbf{x}$ with the initial condition $\mathbf{x}(0) = (1, 4)^T$.

- We need

$$\mathbf{x}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)$$

  - $C_1 = -5$ and $C_2 = 9$.

- The solution is

$$\begin{aligned}\mathbf{x}(t) &= -5\mathbf{x}_1(t) + 9\mathbf{x}_2(t)\\ &= \begin{pmatrix} 10e^{-2t} - 9e^{2t} \\ -5e^{-2t} + 9e^{2t} \end{pmatrix}.\end{aligned}$$

Homogeneous Systems

$$\mathbf{x}' = A\mathbf{x}$$

Proposition: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_k(t)$ are solutions to the homogeneous system, and c_1, c_2, . . . , and c_k are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is *linearly independent* if they are linearly independent at one value of t.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of t.