Math 211

Lecture #26

Solutions of a Planar System

October 28, 2001

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, which is equal to $\mathrm{null}(A - \lambda I)$.
- If λ is an eigenvalue and $\mathbf{v}$ is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, *if we can.*
  - This must be explored further.

Return

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

where

- ◆ $T = \operatorname{tr} A = a_{11} + a_{22}$ and
- ◆ $D = \det A = a_{11}a_{22} - a_{12}a_{21}$.

- The eigenvalues of A are the roots of $p(\lambda) = \lambda^2 - T\lambda + D$,

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Three cases:
  - 2 distinct real roots if $T^2 - 4D > 0$
  - 2 complex conjugate roots if $T^2 - 4D < 0$
  - Double real root if $T^2 - 4D = 0$

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Complex Eigenvalues

- If the discriminant $T^2 - 4D < 0$ we have complex eigenvalues

$$\lambda = \frac{T + i\sqrt{4D - T^2}}{2}, \quad \overline{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}$$

- Example: $\begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$.
  - Characteristic polynomial: $p(\lambda) = \lambda^2 - 2\lambda + 5$.
  - Eigenvalues: $\lambda = 1 + 2i$ and $\overline{\lambda} = 1 - 2i$
  - Eigenvectors: ?
    - We need to know more about complex numbers and matrices.

Complex Numbers - I

- $z = x + iy$, $x, y \in \mathbf{R}$, $i^2 = -1$.
    - $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.
    - $\overline{z} = x - iy$ is the *complex conjugate* of z.
    - $|z| = \sqrt{x^2 + y^2}$ is the *absolute value* of z.
- Formulas:
    - $x = \mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}, \quad y = \mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}.$
    - $\overline{z + w} = \overline{z} + \overline{w}, \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$
    - $|z|^2 = z\overline{z}, \quad |zw| = |z||w|.$
    - $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}, \quad \dfrac{z}{w} = \dfrac{z\overline{w}}{|w|^2}.$

Complex Numbers - II

- *Euler's formula:* $e^{iy} = \cos y + i \sin y$.
  - $e^{z} = e^{x+iy} = e^{x} e^{iy} = e^{x}[\cos y + i \sin y]$.
  - The complex exponential behaves like the real exponential, so $e^{z+w} = e^{z} \cdot e^{w}$.
  - $|e^{z}| = e^{\mathrm{Re}(z)}, \quad \overline{e^{z}} = e^{\bar{z}}$.
- $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$ even if λ is complex.
- Complex polar coordinates: $z = x + iy = |z| e^{i\theta}$, where $\tan \theta = y/x$.

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Complex Matrices

Matrices (or vectors) with complex entries inherit many of the properties of complex numbers.

- $M = A + iB$ where $A = \mathrm{Re} M$ and $B = \mathrm{Im} M$ are real matrices.
- $\overline{M} = A - iB$; $\quad M = \overline{M} \Leftrightarrow M$ is real.
- $\mathrm{Re} M = \frac{1}{2}(M + \overline{M})$; $\quad \mathrm{Im} M = \frac{1}{2i}(M - \overline{M})$
- $\overline{M + N} = \overline{M} + \overline{N}$
- $\overline{M\mathbf{z}} = \overline{M}\overline{\mathbf{z}}$

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Complex Eigenpairs

A a real matrix

- Suppose that λ is a complex eigenvalue with associated eigenvector $\mathbf{w}$. Then $A\mathbf{w} = \lambda\mathbf{w}$.
- Conjugating we get

$$\overline{A\mathbf{w}} = \overline{A}\,\overline{\mathbf{w}} = A\overline{\mathbf{w}}$$

$$\overline{\lambda\mathbf{w}} = \overline{\lambda}\,\overline{\mathbf{w}}$$

- $A\mathbf{w} = \lambda\mathbf{w} \Rightarrow \overline{A\mathbf{w}} = \overline{\lambda\mathbf{w}} \Rightarrow A\overline{\mathbf{w}} = \overline{\lambda}\,\overline{\mathbf{w}}$
- $\Rightarrow \overline{\lambda}$ is an eigenvalue of A with associated eigenvector $\overline{\mathbf{w}}$

- Thus complex eigenvalues come in conjugate pairs λ and $\overline{\lambda}$.
- The associated eigenvectors also come in conjugate pairs $\mathbf{w}$ and $\overline{\mathbf{w}}$.
- $\lambda \neq \overline{\lambda} \quad \Rightarrow \quad \mathbf{w}$ and $\overline{\mathbf{w}}$ are linearly independent.
- We get complex exponential solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} \quad \text{and} \quad \overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}.$$

- $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent complex valued solutions to $\mathbf{x}' = A\mathbf{x}$.

$$\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t) \quad \& \quad \overline{\mathbf{z}}(t) = \mathbf{x}(t) - i\mathbf{y}(t)$$

$$\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) + \overline{\mathbf{z}}(t)}{2}$$

$$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) - \overline{\mathbf{z}}(t)}{2i}$$

- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real valued solutions.
- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are linearly independent.

Solutions in Our Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

- The eigenvalues are: $\lambda = 1 + 2i$ and $\overline{\lambda} = 1 - 2i$.
- An eigenvector associated to λ is $\mathbf{w} = (3 - i, 1)^T$.
- Complex solutions:

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} = e^{(1+2i)t}\begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$$

$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}} = e^{(1-2i)t}\begin{pmatrix} 3 + i \\ 1 \end{pmatrix}$$

◆ This is a fundamental set of solutions.

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- Real solutions:

$$\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$$

  - This is a fundamental set of solutions.

Initial Value Problem

Solve $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

The solution is

$$\begin{aligned} \mathbf{u}(t) = 3e^t &\begin{pmatrix} 3\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \\ &+ 4e^t \begin{pmatrix} 3\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} \\ = e^t &\begin{pmatrix} 5\cos 2t + 15\sin 2t \\ 3\cos 2t + 4\sin 2t \end{pmatrix} \end{aligned}$$

Summary — Complex Eigenvalues

Suppose A is a real 2×2 matrix with

- complex conjugate eigenvalues λ and $\overline{\lambda}$, and
- associated nonzero eigenvectors $\mathbf{w}$ and $\overline{\mathbf{w}}$.

Then

- $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$ and $\overline{\mathbf{z}}(t) = e^{\overline{\lambda}t}\overline{\mathbf{w}}$ form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t))$ form a real valued fundamental set of solutions.

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Examples

$$\mathbf{x}' = A\mathbf{x}$$

where

- $A = \begin{pmatrix} 7 & 30 \\ -3 & -11 \end{pmatrix}$
- $A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$

Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

Proposition: Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix A, and that $\mathbf{v}_1 \neq 0$ and $\mathbf{v}_2 \neq 0$ are eigenvectors associated with λ_1 and λ_2, respectively. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

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