Math 211

Lecture #27

Planar Systems

October 30, 2002

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If λ is an eigenvalue and $\mathbf{v}$ is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, *if we can.*
    - This must be explored further.

Return

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

where

- ♦ $T = \operatorname{tr} A = a_{11} + a_{22}$ and
- ♦ $D = \det A = a_{11}a_{22} - a_{12}a_{21}$.

- The eigenvalues of A are the roots of $p(\lambda) = \lambda^2 - T\lambda + D$,

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Three cases:
  - 2 distinct real roots if $T^2 - 4D > 0$
  - 2 complex conjugate roots if $T^2 - 4D < 0$
  - Double real root if $T^2 - 4D = 0$

Return

Real Distinct Eigenvalues

Suppose A is a real 2×2 matrix with real eigenvalues $\lambda_1 \neq \lambda_2$, and associated nonzero eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Then $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_1 t}\mathbf{v}_2$ form a fundamental set of solutions.

Return

Cases

Complex Eigenvalues

Suppose A is a real 2×2 matrix with complex conjugate eigenvalues λ and $\overline{\lambda}$, and associated nonzero eigenvectors $\mathbf{w}$ and $\overline{\mathbf{w}}$.

Then

- $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$ and $\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}$ form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t))$ form a real valued fundamental set of solutions.

Examples

$$\mathbf{x}' = A\mathbf{x}$$

where

- $A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$
- $A = \begin{pmatrix} 7 & 30 \\ -3 & -11 \end{pmatrix}$

Double Real Root

In this case $T^2 - 4D = 0$.

- There is only one eigenvalue

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T}{2}.$$

- The eigenspace of λ has dimension 1 or 2.
  - If the dimension is 2, then $A = \lambda I$.
  - Every vector is an eigenvector. Every solution has the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}.$$

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2; \quad \lambda = -2$
- $A - \lambda I = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}; \quad \mathbf{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
- The eigenspace has dimension 1, with basis $\mathbf{v}$.
- The standard procedure yields only one solution, $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v} = e^{-2t}(-3\,1)^T$.

Second Solution

- Look for a second solution of the form

$$\mathbf{x}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1]$$

Then $\mathbf{x}_2' = e^{\lambda t}[(\mathbf{v}_1 + \lambda\mathbf{v}_2) + \lambda t\mathbf{v}_1]$

$$A\mathbf{x}_2 = e^{\lambda t}[A\mathbf{v}_2 + tA\mathbf{v}_1]$$

- $\mathbf{x}_2' = A\mathbf{x}_2 \Leftrightarrow A\mathbf{v}_1 = \lambda\mathbf{v}_1$ and

$$A\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2.$$

- We need two things:
  - $\mathbf{v}_1$ must be an eigenvector.
  - $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$.

Return

The Degenerate Planar Case

- Find the (only) eigenvalue λ_1.
- Find an eigenvector $\mathbf{v}_1 \neq \mathbf{0}$.
- Find $\mathbf{v}_2$ with $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$. To do so:
  - Start with any vector $\mathbf{w}$ not a multiple of $\mathbf{v}_1$
  - Then $(A - \lambda I)\mathbf{w} = a\mathbf{v}_1$ with $a \neq 0$.
  - Set $\mathbf{v}_2 = \frac{1}{a}\mathbf{w}$. $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$.
- $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1]$ form a fundamental set of solutions.

Example (cont.)

- Start with $\mathbf{w} = (1,0)^T$.
- $\mathbf{v}_2 = -\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
- Fundamental set of solutions:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 = e^{-2t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{x}_2(t) &= e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1] \\ &= e^{-2t}\begin{pmatrix} -1-3t \\ t \end{pmatrix}. \end{aligned}$$

Examples

Solve $\mathbf{x}' = A\mathbf{x}$, where

- $$A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

- $$A = \begin{pmatrix} 0 & 9 \\ -1 & -6 \end{pmatrix}$$

Planar System $\mathbf{x}' = A\mathbf{x}$

- Equilibrium points for the system
  - Set of equilibrium points equals $\mathrm{null}(A)$.
  - If A nonsingular the only equilibrium point is $\mathbf{0}$.
- Can we list the types of all possible equilibrium points for planar linear systems?
  - Six most important cases.
  - Look at solution curves in the phase plane.

Return

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The general solution is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- There are three subcases depending on the signs of the eigenvalues.

Return Cases

Exponential Solutions

$$\mathbf{x}(t) = Ce^{\lambda t}\mathbf{v}$$

- The solution curve is a straight half-line through $C\mathbf{v}$. Sometimes called *half-line* solutions.
- If $\lambda > 0$ the solution starts at $\mathbf{0}$ for $t = -\infty$, and tends to ∞ as $t \to \infty$. *Unstable solution*
- If $\lambda < 0$ the solution starts at ∞ for $t = -\infty$, and tends to $\mathbf{0}$ as $t \to \infty$. *Stable solution*

Return

Real case

Saddle Point

- $\lambda_1 < 0 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Two stable exponential solutions $(C_2 = 0)$
- Two unstable exponential solutions $(C_1 = 0)$.
- $C_1 \neq 0$ and $C_2 \neq 0$.
  - As $t \to \infty$, $\mathbf{x}(t) \to \infty$, approaching the half-line through $C_2\mathbf{v}_2$.
  - As $t \to -\infty$, $\mathbf{x}(t) \to \infty$, approaching the half-line through $C_2\mathbf{v}_1$.

Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Four stable exponential solutions.
- All solutions $\rightarrow \mathbf{0}$ as $t \rightarrow \infty$. (Stable)
  - Tangent to $C_2\mathbf{v}_2$ if $C_2 \neq 0$.
- All solutions $\rightarrow \infty$ as $t \rightarrow -\infty$.
  - $\|$ to the half line through $C_1\mathbf{v}_1$ if $C_1 \neq 0$.

Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Four unstable exponential solutions.
- All solutions $\rightarrow \mathbf{0}$ as $t \rightarrow -\infty$.
    - ♦ Tangent to $C_1\mathbf{v}_1$ if $C_1 \neq 0$.
- All solutions $\rightarrow \infty$ as $t \rightarrow \infty$. (Unstable)
    - ♦ $\|$ to the half line through $C_2\mathbf{v}_2$ if $C_2 \neq 0$.

Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

Proposition: Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix A, and that $\mathbf{v}_1 \neq 0$ and $\mathbf{v}_2 \neq 0$ are eigenvectors associated with λ_1 and λ_2, respectively. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Return