Math 211

Lecture #29

Phase Plane Portraits

Systems of Higher Dimension

November 4, 2002

Planar System $\mathbf{x}' = A\mathbf{x}$

- Equilibrium points for the system
  - Set of equilibrium points equals $\mathrm{null}(A)$.
  - A nonsingular $\Rightarrow$ only equilibrium point is $\mathbf{0}$.
- Can we list the types of all possible equilibrium points for planar linear systems?
  - We will do the six most important cases.
  - Look at solution curves in the phase plane.

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- $\lambda_1 < 0 < \lambda_2$ Saddle point.
- $\lambda_1 < \lambda_2 < 0$ Nodal sink.
- $0 < \lambda_1 < \lambda_2$ Nodal source.

Complex Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D < 0$

$$\lambda = \alpha + i\beta \quad \text{and} \quad \bar{\lambda} = \alpha - i\beta.$$

- Eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ associated to λ.
- General solution

$$\begin{aligned}\mathbf{x}(t) = C_1 e^{\alpha t}&[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] \\ &+ C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]\end{aligned}$$

- $\alpha = \mathrm{Re}(\lambda) = 0$ Center.
- $\alpha = \mathrm{Re}(\lambda) < 0$ Spiral sink.
- $\alpha = \mathrm{Re}(\lambda) > 0$ Spiral source.

Planar Systems

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- The characteristic polynomial is $p(\lambda) = \lambda^2 - T\lambda + D$. where
  - ♦ $T = \operatorname{tr} A = a_{11} + a_{22}$ and
  - ♦ $D = \det A = a_{11}a_{22} - a_{12}a_{21}$.
- The eigenvalues are

$$\lambda_1,\ \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- $\lambda_1 \,\&\, \lambda_2$ are the roots of $p(\lambda) = \lambda^2 - T\lambda + D$, so

$$\begin{aligned} p(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{aligned}$$

- Hence, $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$.
- Duality between $(\lambda_1,\ \lambda_2)$ and (T, D).
- We will represent a system by the location of (T, D) in the TD-plane — the *trace-determinant plane.*

Trace-Determinant Plane

- $T^2 - 4D > 0$
  - $\Rightarrow$ distinct real eigenvalues $\lambda_1 \,\&\, \lambda_2$
  - $D = \lambda_1\lambda_2 < 0 \Rightarrow$ Saddle point.
  - $D = \lambda_1\lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
    - $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
    - $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.

- $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues

$$\lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta.$$

  - $T = \lambda + \overline{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source.
  - $T = \lambda + \overline{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink.
  - $T = \lambda + \overline{\lambda} = 2\alpha = 0 \Rightarrow$ Center.

Types of Equilibrium Points

- *Generic* types
  - Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.
- *Nongeneric* types
  - Center and many others. Occupy pieces of the boundaries between the generic types.

Return

Higher Dimensional Systems

$$\mathbf{x}' = A\mathbf{x}$$

- A is a real $n \times n$ matrix.
- If λ is an eigenvalue and $\mathbf{v} \neq 0$ is an associated eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Much like the planar case, but now we need n linearly independent solutions.
- We no longer have the easy way to compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

Return

Proposition: Suppose that $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A, and that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are associated nonzero eigenvectors. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Theorem: Suppose the $n \times n$ real matrix A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are associated nonzero eigenvectors. Then the exponential solutions $\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i$, $1 \leq i \leq n$ form a fundamental set of solutions for the system $\mathbf{x}' = A\mathbf{x}$.

Examples:

- $A = \begin{pmatrix} -2 & 3 & -4 \\ 0 & 1 & 0 \\ 0 & 4 & -1 \end{pmatrix}$
- $A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix}$
    - Use MATLAB.

Complex Eigenvalues

A a real $n \times n$ matrix with a complex eigenvalue λ and associate eigenvector $\mathbf{w}$.

- $\Rightarrow$ $\overline{\lambda}$ is an eigenvalue and $\overline{\mathbf{w}}$ is an associated nonzero eigenvector.
- Complex valued solutions: $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$

$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}.$$

- Real solutions: $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$

$$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)).$$

Return

Example

$$A = \begin{pmatrix} 21 & 10 & 4 \\ -70 & -31 & -10 \\ 30 & 10 & -1 \end{pmatrix}$$

- The theorem applies if some of the eigenvalues are complex and we replace complex conjugate pairs of solutions by their real and imaginary parts.

Repeated Eigenvalues – Example 1

$$A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
  - Eigenspace has dimension 1 $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{-3t}(-1/3, 2/3, 1)^T$$

- $\lambda_2 = -1$
  - Eigenspace has dimension 2 $\Rightarrow$ two linearly independent exponential solutions
  - Eigenspace has basis $\mathbf{v}_2 = (-5/2, 1, 0)^T$ and $\mathbf{v}_3 = (3/2, 0, 1)^T$.
  - Linearly independent solutions

$$\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -5/2 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad \mathbf{x}_3(t) = e^{-t}\begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}$$

- $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are a fundamental set of solutions.

Repeated Eigenvalues – Example 2

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
  - Eigenspace has dimension 1 $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{-3t}(-1/2, 3/2, 1)^T$$

- $\lambda_2 = -1$
  - Eigenspace has dimension 1 $\Rightarrow$ only one exponential solution

$$\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

- Need a third solution.
- Need a new idea.

Multiplicities

A an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of λ_j is q_j.
- The *geometric multiplicity* of λ_j is d_j, the dimension of the eigenspace of λ_j.

- We always have:
  - $q_1 + q_2 + \cdots + q_k = n$.
  - $1 \leq d_j \leq q_j$.
  - There are d_j linearly independent exponential solutions corresponding to λ_j.
  - If $d_j = q_j$ for all j we have n linearly independent solutions.
- If $d_j < q_j$ we have trouble.