Math 211

Lecture #31

Exponential of a Matrix
Stability of Solutions

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Exponential of a Matrix

Definition: The *exponential* of the $n \times n$ matrix A is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_0^\infty \frac{1}{n!}A^n.$$

Theorem: The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is $\mathbf{x}(t) = e^{tA}\mathbf{v}$.

- Can we compute $e^{tA}\mathbf{v}$ for enough vectors to find a fundamental set of solutions?

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Key to Computing e^{tA} or $e^{tA}\mathbf{v}$

Suppose that A an $n \times n$ matrix, and λ a number (an eigenvalue). Then

$$e^{tA} = e^{\lambda t} \cdot [I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots]$$
$$e^{tA}\mathbf{v} = e^{\lambda t} \cdot [\mathbf{v} + t(A - \lambda I)\mathbf{v} + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v} + \cdots]$$

- If λ is an eigenvalue and $\mathbf{v}$ is an associated eigenvector, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
- If $(A - \lambda I)^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}[\mathbf{v} + t(A - \lambda I)\mathbf{v}]$.

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Example 2, Reprise

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda+3)(\lambda+1)^2$
- $\lambda_1 = -3$, with algebraic multiplicity 1.
  - $\mathrm{null}(A - \lambda_1 I)$ has basis $\mathbf{v}_1 = (-1/2, 3/2, 1)^T$, so the geometric multiplicity is 1.
  - There is one exponential solution

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-3t}(-1/2, 3/2, 1)^T.$$

- $\lambda_2 = -1$, with algebraic multiplicity 2.
  - $\mathrm{null}(A - \lambda_2 I)$ has basis $\mathbf{v}_2 = (-1/2, 1, 1)^T$, so the geometric multiplicity is 1.
  - So there is only one exponential solution

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{-t}(-1/2, 1, 1)^T.$$

  - However, $\mathrm{null}((A - \lambda_2 I)^2)$ has dimension 2, with basis $(0, 1, 1)^T)$ and $(1, 0, 0)^T$. With $\mathbf{v}_3 = (1, 0, 0)^T$ we get the third solution

$$\begin{aligned} \mathbf{x}_3(t) &= e^{tA}\mathbf{v}_3 = e^{-t}[\mathbf{v}_3 + t(A+I)\mathbf{v}_3] \\ &= e^{-t}(1 + 2t, -4t, -4t)^T. \end{aligned}$$

- $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are a fundamental set of solutions.

Generalized Eigenvectors

Definition: If λ is an eigenvalue of A and $(A - \lambda I)^p\mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a *generalized eigenvector* associated with λ.

- Then

$$\begin{aligned} e^{tA}\mathbf{v} = e^{\lambda t}\Bigg[&\mathbf{v} + +t(A - \lambda I)\mathbf{v} + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v} \\ &+ \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v}\Bigg] \end{aligned}$$

- We can compute $e^{tA}\mathbf{v}$ for any generalized eigenvector.

Solution Strategy

Theorem: If λ is an eigenvalue of A with algebraic multiplicity q, then there is an integer $p \leq q$ such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.

- Thus, we can find q linearly independent solutions associated with the eigenvalue λ.
- Since the sum of the algebraic multiplicities is n, we can find a fundamental set of solutions.

Procedure for Solving $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A.
- For each eigenvalue λ:
  - Find the algebraic multiplicity q.
  - Find the smallest integer p such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.
  - Find a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ of $\mathrm{null}((A - \lambda I)^p)$.
  - For $j = 1, 2, \ldots, q$, set $\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j$.
  - If λ is complex, find real solutions.

Examples

- Use MATLAB.

Procedure for a Complex Eigenvalue

If λ is a complex eigenvalue of algebraic multiplicity q.
Then $\overline{\lambda}$ also has algebraic multiplicity q.

- Find the smallest integer p such that $\mathrm{null}((A-\lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{w}_1,\, \mathbf{w}_2,\, \dots,\, \mathbf{w}_q$ of $\mathrm{null}((A-\lambda I)^p)$.
- For $j = 1, 2, \dots, q$, set $\mathbf{z}_j(t) = e^{tA}\mathbf{w}_j$. $\mathbf{z}_1,\, \dots,\, \mathbf{z}_q$. Together with $\overline{\mathbf{z}_1},\, \dots,\, \overline{\mathbf{z}_q}$, these are $2q$ linearly independent complex valued solutions.
- For $j = 1, 2, \dots, q$, set $\mathbf{x}_j(t) = \mathrm{Re}(\mathbf{z}_j(t))$ and $\mathbf{y}_j(t) = \mathrm{Im}(\mathbf{z}_j(t))$. These are $2q$ linearly independent real valued solutions.

Stability

Autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with an equilibrium point at $\mathbf{x}_0$.

- Basic question: What happens to *all solutions* as $t \to \infty$?
- $\mathbf{x}_0$ is *stable* if for every $\epsilon > 0$ there is a $\delta > 0$ such that a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta \Rightarrow |\mathbf{x}(t) - \mathbf{x}_0| < \epsilon$ for all $t \geq 0$.
  - Every solution that starts close to $\mathbf{x}_0$ stays close to $\mathbf{x}_0$.

- $\mathbf{x}_0$ is *asymptotically stable* if it is stable and there is an $\eta > 0$ such that if $\mathbf{x}(t)$ is a solution with $|\mathbf{x}(0) - \mathbf{x}_0| < \eta$, then $\mathbf{x}(t) \to \mathbf{x}_0$ as $t \to \infty$.
  - $\mathbf{x}_0$ is called a *sink*.
  - Every solution that starts close to $\mathbf{x}_0$ approaches $\mathbf{x}_0$.
- $\mathbf{x}_0$ is *unstable* if there is an $\epsilon > 0$ such that for any $\delta > 0$ there is a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$ with the property that there are values of $t > 0$ such that $|\mathbf{x}(t) - \mathbf{x}_0| > \epsilon$.
  - There are solutions starting arbitrarily close to $\mathbf{x}_0$ that move away from $\mathbf{x}_0$.

Examples $D = 2$

- Sinks are asymptotically stable.
  - The eigenvalues have negative real part.
- Sources are unstable.
  - The eigenvalues have positive real part.
- Saddles are unstable.
  - One eigenvalue has positive real part.
- Centers are stable but not asymptotically stable.
  - The eigenvalues have real part $= 0$.

Theorem: Let A be an $n \times n$ real matrix.

- Suppose the real part of every eigenvalue of A is negative. Then $\mathbf{0}$ is an asymptotically stable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.
- Suppose A has at least one eigenvalue with positive real part. Then $\mathbf{0}$ is an unstable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.

Examples

- $D = 2$
  - $T^2 - 4D = 0$.
    - $T < 0 \Rightarrow$ sink. $T > 0 \Rightarrow$ source.
- $\mathbf{y}' = A\mathbf{y}$,

$$A = \begin{pmatrix} -2 & -18 & -7 & -14 \\ 1 & 6 & 2 & 5 \\ 2 & 2 & -3 & 0 \\ -2 & -8 & -1 & -6 \end{pmatrix}.$$

  - A has eigenvalues -1, -2, & $-1 \pm i$.
  - $\mathbf{0}$ is asymptotically stable.

Multiplicities

A an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.

- The characteristic polynomial has the form

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of λ_j is q_j.
- The *geometric multiplicity* of λ_j is d_j, the dimension of the eigenspace of λ_j.
- $q_1 + q_2 + \ldots + q_k = n$.
- $1 \le d_j \le q_j$.

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