Math 211

Lecture #32

Higher Order Equations
Harmonic Motion

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Higher Order Equations

- Linear homogenous equation of order n.

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

- Linear homogenous equation of second order.

$$y'' + py' + qy = 0$$

- Equivalent system: $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

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Linear Independence

- A fundamental set of solutions for the system consists of two linearly independent solutions.

Definition: Two functions $u(t)$ and $v(t)$ are *linearly independent* if neither is a constant multiple of the other.

- $u(t)$ and $v(t)$ are linearly independent solutions to $y'' + py' + qy = 0 \Leftrightarrow \begin{pmatrix} u \\ u' \end{pmatrix}$ & $\begin{pmatrix} v \\ v' \end{pmatrix}$ are linearly independent solutions to the equivalent system.

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General Solution

Theorem: Suppose that $y_1(t)$ & $y_2(t)$ are linearly independent solutions to the equation

$$y'' + py' + qy = 0.$$

Then the general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

Definition: A set of two linearly independent solutions is called a *fundamental set of solutions.*

Solutions to $y'' + py' + qy = 0.$

- The equivalent system has exponential solutions.
- Look for exponential solutions to the 2nd order equation of the form $y(t) = e^{\lambda t}$.
- *Characteristic equation:* $\lambda^2 + p\lambda + q = 0.$
  - *Characteristic polynomial:* $\lambda^2 + p\lambda + q.$
  - Same for the 2^{nd} order equation and the system.

Real Roots

- If λ is a root to the characteristic polynomial then $y(t) = e^{\lambda t}$ is a solution.
  - If the characteristic polynomial has two distinct real roots λ_1 and λ_2, then $y(_1 t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are a fundamental set of solutions.
- If λ is a root to the characteristic polynomial of multiplicity 2, then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ are a fundamental set of solutions.

Complex Roots

- If $\lambda = \alpha + i\beta$ is a complex root of the characteristic equation, then so is $\overline{\lambda} = \alpha - i\beta$.
- A complex valued fundamental set of solutions is

$$z(t) = e^{\lambda t} \quad \text{and} \quad \overline{z}(t) = e^{\overline{\lambda} t}.$$

- A real valued fundamental set of solutions is

$$x(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = e^{\alpha t} \sin \beta t.$$

Examples

- $y'' - 5y' + 6y = 0$, with $y(0) = 0$ and $y'(0) = 1$.
- $y'' + 4y' + 13y = 0$, with $y(0) = -1$ and $y'(0) = 14$.
- $y'' + 4y' + 4y = 0$, with $y(0) = 2$ and $y'(0) = 0$.
- $y'' + 25y = 0$, with $y(0) = 3$ and $y'(0) = -2$.

The Vibrating Spring

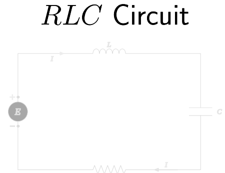
Newton's second law: $ma =$ total force.

- Forces acting:
  - Gravity mg.
  - Restoring force $R(x)$.
  - Damping force $D(v)$.
  - External force $F(t)$.

- Including all of the forces, Newton's law becomes
$$ma = mg + R(x) + D(v) + F(t)$$
- Hooke's law: $R(x) = -kx$.
    - $k > 0$ is the *spring constant*.
    - Spring-mass equilibrium $x_0 = mg/k$.
    - Set $y = x - x_0$. Newton's law becomes
$$my'' = -ky + D(y') + F(t).$$

- Damping force $D(y') = -\mu y'$.
    - $\mu \geq 0$ is the *damping constant*.
    - Newton's law becomes
$$my'' = -ky - \mu y' + F(t), \quad \text{or}$$
$$my'' + \mu y' + ky = F(t), \quad \text{or}$$
$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t).$$
- This is the equation of the vibrating spring.

RLC Circuit



$$LI'' + RI' + \frac{1}{C}I = E'(t), \quad \text{or}$$
$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t).$$
- This is the equation of the RLC circuit.

Harmonic Motion

- Spring: $y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t)$.
- Circuit: $I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{1}{L} E'(t)$.
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

  - We call this the equation for *harmonic motion*.
  - It includes both the vibrating spring and the RLC circuit.

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The Equation for Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- ω_0 is the *natural frequency.*
  - Spring: $\omega_0 = \sqrt{k/m}$.
  - Circuit: $\omega_0 = \sqrt{1/LC}$.
- c is the *damping constant.*
  - Spring: $2c = \mu/m$.
  - Circuit: $2c = R/L$.
- $f(t)$ is the *forcing term.*

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Simple Harmonic Motion

No forcing , and no damping.

$$x'' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + \omega_0^2$, $\lambda = \pm i\omega_0$.
- Fundamental set of solutions: $x_1(t) = \cos \omega_0 t$ & $x_2(t) = \sin \omega_0 t$.
- General solution: $x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$.
- Every solution is periodic with the natural frequency ω_0.
  - The period is $T = 2\pi/\omega_0$.

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Amplitude and Phase

- Put C_1 and C_2 in polar coordinates:

$$C_1 = A\cos\phi, \ \& \ C_2 = A\sin\phi.$$

- Then $x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

$$= A\cos(\omega_0 t - \phi).$$

- A is the *amplitude*; $A = \sqrt{C_1^2 + C_2^2}$.
- ϕ is the *phase*; $\tan\phi = C_2/C_1$.

Examples

- $C_1 = 3$, $C_2 = 4 \ \Rightarrow \ A = 5$, $\phi = 0.9273$.
- $C_1 = -3$, $C_2 = 4 \ \Rightarrow \ A = 5$, $\phi = 2.2143$.
- $C_1 = -3$, $C_2 = -4 \ \Rightarrow \ A = 5$, $\phi = -2.2143$.

Example

$$x'' + 16x = 0, \ x(0) = -2 \ \& \ x'(0) = 4$$

- Natural frequency: $\omega_0^2 = 16 \ \Rightarrow \ \omega_0 = 4$.
- General solution: $x(t) = C_1 \cos 4t + C_2 \sin 4t$.
- IC: $-2 = x(0) = C_1$, and $4 = x'(0) = 4C_2$.
- Solution

$$x(t) = -2\cos 2t + \sin 2t$$
$$= \sqrt{5}\cos(2t - 2.6779).$$