

Math 211

Lecture #33

Harmonic Motion
Inhomogeneous Equations

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Harmonic Motion

- Spring: $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$.
- Circuit: $I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t)$.
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2x = f(t).$$

- We call this the equation for *harmonic motion*.
- It includes both the vibrating spring and the *RLC* circuit.

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The Equation for Harmonic Motion

$$x'' + 2cx' + \omega_0^2x = f(t).$$

- ω_0 is the *natural frequency*.
 - Spring: $\omega_0 = \sqrt{k/m}$.
 - Circuit: $\omega_0 = \sqrt{1/LC}$.
- c is the *damping constant*.
 - Spring: $2c = \mu/m$.
 - Circuit: $2c = R/L$.
- $f(t)$ is the *forcing term*.

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Simple Harmonic Motion

No forcing, and no damping.

$$x'' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + \omega_0^2$, $\lambda = \pm i\omega_0$.
- Fundamental set of solutions: $x_1(t) = \cos \omega_0 t$ & $x_2(t) = \sin \omega_0 t$.
- General solution: $x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$.
- Every solution is periodic at the natural frequency ω_0 .
 - ♦ The period is $T = 2\pi/\omega_0$.

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Amplitude and Phase

- Put C_1 and C_2 in polar coordinates:

$$C_1 = A \cos \phi, \text{ \& } C_2 = A \sin \phi.$$

- Then $x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$
 $= A \cos(\omega_0 t - \phi)$.
- A is the *amplitude*; $A = \sqrt{C_1^2 + C_2^2}$.
- ϕ is the *phase*; $\tan \phi = C_2/C_1$.

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Examples

- $C_1 = 3, C_2 = 4 \Rightarrow A = 5, \phi = 0.9273$.
- $C_1 = -3, C_2 = 4 \Rightarrow A = 5, \phi = 2.2143$.
- $C_1 = -3, C_2 = -4 \Rightarrow A = 5, \phi = -2.2143$.

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Amplitude & phase

Example

$$x'' + 16x = 0, x(0) = -2 \text{ \& } x'(0) = 4$$

- Natural frequency: $\omega_0^2 = 16 \Rightarrow \omega_0 = 4$.
- General solution: $x(t) = C_1 \cos 4t + C_2 \sin 4t$.
- IC: $-2 = x(0) = C_1$, and $4 = x'(0) = 4C_2$.
- Solution

$$\begin{aligned} x(t) &= -2 \cos 2t + \sin 2t \\ &= \sqrt{5} \cos(2t - 2.6779). \end{aligned}$$

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Amplitude & phase

Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$; roots $-c \pm \sqrt{c^2 - \omega_0^2}$.
- Three cases
 - ♦ $c < \omega_0$ — *underdamped case*
 - ♦ $c > \omega_0$ — *overdamped case*
 - ♦ $c = \omega_0$ — *critically damped case*

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Harmonic motion

Underdamped Case

- $c < \omega_0$
- Two complex roots λ and $\bar{\lambda}$, where $\lambda = -c + i\omega$ and $\omega = \sqrt{\omega_0^2 - c^2}$.
- General solution

$$\begin{aligned} x(t) &= e^{-ct} [C_1 \cos \omega t + C_2 \sin \omega t] \\ &= A e^{-ct} \cos(\omega t - \phi) \end{aligned}$$

Overdamped Case

- $c > \omega_0$, so two real roots

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$$

$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

- $\lambda_1 < \lambda_2 < 0$.
- General solution

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

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Critically Damped Case

- $c = \omega_0$
- One negative real root $\lambda = -c$ with multiplicity 2.
- General solution

$$x(t) = e^{-ct}[C_1 + C_2 t].$$

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Inhomogeneous Equations

$$y'' + py' + qy = f(t)$$

- The corresponding homogeneous equation is

$$y'' + py' + qy = 0$$

- We know how to find a fundamental set of solutions y_1 and y_2 .
- The general solution of the homogeneous equation is $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$.

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Theorem: Assume

- $y_p(t)$ is a particular solution to the inhomogeneous equation $y'' + py' + qy = f(t)$;
- $y_1(t)$ & $y_2(t)$ is a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$.

Then the general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + C_1y_1(t) + C_2y_2(t).$$

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[Homogeneous equation](#)

Method of Undetermined Coefficients

$$y'' + py' + qy = f(t)$$

- If the forcing term $f(t)$ has a form which is replicated under differentiation, then look for a particular solution of the same general form as the forcing term.

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Exponential Forcing Term

$$y'' + py' + qy = Ce^{at}$$

- Example: $y'' + 3y' + 2y = 4e^{-3t}$
- Try $y_p(t) = ae^{-3t}$; a to be determined.
 - ♦ Particular solution: $y_p(t) = 2e^{-3t}$.
- Homogeneous equation: $y'' + 3y' + 2y = 0$.
 - ♦ Fundamental set of solutions: e^{-2t} & e^{-t} .
- General solution to the inhomogeneous equation:

$$y(t) = 2e^{-3t} + C_1e^{-t} + C_2e^{-2t}.$$

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Trigonometric Forcing Term

$$y'' + py' + qy = A \cos \omega t + B \sin \omega t$$

- Example: $y'' + 4y' + 5y = 4 \cos 2t - 3 \sin 2t$
- Try $y_p(t) = a \cos 2t + b \sin 2t$
 - ♦ Particular solution: $y_p(t) = [28 \cos 2t + 29 \sin 2t]/65$.
- Homogeneous equation: $y'' + 4y' + 5y = 0$
 - ♦ Fund. set of sol'ns: $e^{-2t} \cos t$ & $e^{-2t} \sin t$.
- General solution to the inhomogeneous equation:

$$y(t) = \frac{28 \cos 2t + 29 \sin 2t}{65} + e^{-2t}[C_1 \cos t + C_2 \sin t].$$

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Complex Method

$$x'' + px' + qx = A \cos \omega t \quad \text{or}$$

$$y'' + py' + qy = A \sin \omega t.$$

- Solve $z'' + pz' + qz = Ae^{i\omega t}$.
 - ♦ Try $z(t) = ae^{i\omega t}$.
- $x_p(t) = \operatorname{Re}(z(t))$ and $y_p(t) = \operatorname{Im}(z(t))$.

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Example

$$x'' + 4x' + 5x = 4 \cos 2t$$

- Solve $z'' + 4z' + 5z = 4e^{2it}$.
 - ♦ Try $z(t) = ae^{2it}$.
 - ♦ Particular solution: $z(t) = (4 - 32i)e^{2it}/65$.
- Particular solution to the real equation:

$$\begin{aligned} x_p(t) &= \operatorname{Re}(z(t)) \\ &= [4 \cos 2t + 32 \sin 2t]/65. \end{aligned}$$

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Polynomial Forcing Term

$$y'' + py' + qy = P(t)$$

- Example: $y'' - 3y' + 2y = 1 - 4t$.
 - ♦ Try $y(t) = a + bt$.
 - ♦ Particular solution: $y(t) = -5 - 2t$.
- General solution

$$y(t) = -5 - 2t + C_1e^t + C_2e^{2t}.$$

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Exceptional Cases

- Example: $y'' - 3y' + 2y = 3e^t$.
 - ♦ Try $y(t) = ae^t$
 - ♦ The method does not work because e^t is a solution to the associated homogeneous equation.
- Try $y(t) = ate^t$
 - ♦ Particular solution: $y_p(t) = -3te^t$.
- General solution: $y(t) = -3te^t + C_1e^t + C_2e^{2t}$.
- If the suggested particular solution does not work, multiply it by t and try again.

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Combination Forcing Term

Example $y'' + 5y' + 6y = 2e^{2t} - 5 \cos t$

- Solve

$$y_1'' + 5y_1' + 6y_1 = 2e^{2t}$$

$$y_2'' + 5y_2' + 6y_2 = -5 \cos t$$

- Set $y(t) = y_1(t) + y_2(t)$.

[Theorem](#)

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