# Math 211

Lecture #33

Harmonic Motion
Inhomogeneous Equations

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# Harmonic Motion

- Spring: $y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t)$.
- Circuit: $I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{1}{L} E'(t)$.
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

  - We call this the equation for *harmonic motion.*
  - It includes both the vibrating spring and the $RLC$ circuit.

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# The Equation for Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- $\omega_0$ is the *natural frequency.*
  - Spring: $\omega_0 = \sqrt{k/m}$.
  - Circuit: $\omega_0 = \sqrt{1/LC}$.
- $c$ is the *damping constant.*
  - Spring: $2c = \mu/m$.
  - Circuit: $2c = R/L$.
- $f(t)$ is the *forcing term.*

# Simple Harmonic Motion

No forcing , and no damping.

$$x'' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + \omega_0^2$, $\lambda = \pm i\omega_0$.
- Fundamental set of solutions: $x_1(t) = \cos \omega_0 t$ & $x_2(t) = \sin \omega_0 t$.
- General solution: $x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$.
- Every solution is periodic at the natural frequency $\omega_0$.
  - The period is $T = 2\pi/\omega_0$.

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# Amplitude and Phase

- Put $C_1$ and $C_2$ in polar coordinates:

$$C_1 = A\cos\phi, \ \& \, C_2 = A\sin\phi.$$

- Then $x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

$$= A\cos(\omega_0 t - \phi).$$

- $A$ is the *amplitude*; $A = \sqrt{C_1^2 + C_2^2}$.
- $\phi$ is the *phase*; $\tan\phi = C_2/C_1$.

# Examples

- $C_1 = 3,\ C_2 = 4 \ \Rightarrow\ A = 5,\ \phi = 0.9273.$
- $C_1 = -3,\ C_2 = 4 \ \Rightarrow\ A = 5,\ \phi = 2.2143.$
- $C_1 = -3,\ C_2 = -4 \ \Rightarrow\ A = 5,\ \phi = -2.2143.$

# Example

$$x'' + 16x = 0,\ x(0) = -2\ \&\_x'(0) = 4$$

- Natural frequency: $\omega_0^2 = 16 \ \Rightarrow\ \omega_0 = 4.$
- General solution: $x(t) = C_1 \cos 4t + C_2 \sin 4t.$
- IC: $-2 = x(0) = C_1$, and $4 = x'(0) = 4C_2$.
- Solution

$$\begin{aligned} x(t) &= -2\cos 2t + \sin 2t \\ &= \sqrt{5}\cos(2t - 2.6779). \end{aligned}$$

# Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$; roots $-c \pm \sqrt{c^2 - \omega_0^2}$.
- Three cases
  - $c < \omega_0$ — *underdamped case*
  - $c > \omega_0$ — *overdamped case*
  - $c = \omega_0$ — *critically damped case*

# Underdamped Case

- $c < \omega_0$
- Two complex roots $\lambda$ and $\bar{\lambda}$, where $\lambda = -c + i\omega$ and $\omega = \sqrt{\omega_0^2 - c^2}$ .
- General solution

$$\begin{aligned} x(t) &= e^{-ct}[C_1 \cos \omega t + C_2 \sin \omega t] \\ &= A e^{-ct} \cos(\omega t - \phi) \end{aligned}.$$

# Overdamped Case

- $c > \omega_0$ , so two real roots

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$$
$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

- $\lambda_1 < \lambda_2 < 0.$
- General solution

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

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# Critically Damped Case

- $c = \omega_0$
- One negative real root $\lambda = -c$ with multiplicity 2.
- General solution

$$x(t) = e^{-ct}[C_1 + C_2 t].$$

# Inhomogeneous Equations

$$y'' + py' + qy = f(t)$$

- The corresponding homogeneous equation is

$$y'' + py' + qy = 0$$

  - We know how to find a fundamental set of solutions $y_1$ and $y_2$.
  - The general solution of the homogeneous equation is $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$.

**Theorem:** Assume

- $y_p(t)$ is a particular solution to the inhomogeneous equation $y'' + py' + qy = f(t)$;
- $y_1(t)$ & $y_2(t)$ is a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$.

Then the general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t).$$

# Method of Undetermined Coefficients

$$y'' + py' + qy = f(t)$$

- If the forcing term $f(t)$ has a form which is replicated under differentiation, then look for a particular solution of the same general form as the forcing term.

# Exponential Forcing Term

$$y'' + py' + qy = Ce^{at}$$

- Example: $y'' + 3y' + 2y = 4e^{-3t}$
- Try $y_p(t) = ae^{-3t}$; $a$ to be determined.
  - Particular solution: $y_p(t) = 2e^{-3t}$.
- Homogeneous equation: $y'' + 3y' + 2y = 0$.
  - Fundamental set of solutions: $e^{-2t}$ & $e^{-t}$.
- General solution to the inhomogeneous equation:

$$y(t) = 2e^{-3t} + C_1e^{-t} + C_2e^{-2t}.$$

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# Trigonometric Forcing Term

$$y'' + py' + qy = A\cos\omega t + B\sin\omega t$$

- Example: $y'' + 4y' + 5y = 4\cos 2t - 3\sin 2t$
- Try $y_p(t) = a\cos 2t + b\sin 2t$
  - Particular solution: $y_p(t) = [28\cos 2t + 29\sin 2t]/65.$
- Homogeneous equation: $y'' + 4y' + 5y = 0$
  - Fund. set of sol'ns: $e^{-2t}\cos t$ & $e^{-2t}\sin t$.
- General solution to the inhomogeneous equation:

$$y(t) = \frac{28\cos 2t + 29\sin 2t}{65} + e^{-2t}[C_1\cos t + C_2\sin t].$$

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# Complex Method

$$x'' + px' + qx = A\cos\omega t \quad \text{or}$$

$$y'' + py' + qy = A\sin\omega t.$$

- Solve $z'' + pz' + qz = Ae^{i\omega t}$.
  - Try $z(t) = ae^{i\omega t}$.
- $x_p(t) = \mathrm{Re}(z(t))$ and $y_p(t) = \mathrm{Im}(z(t))$.

# Example

$$x'' + 4x' + 5x = 4\cos 2t$$

- Solve $z'' + 4z' + 5z = 4e^{2it}$.
  - Try $z(t) = ae^{2it}$.
  - Particular solution: $z(t) = (4 - 32i)e^{2it}/65$.
- Particular solution to the real equation:

$$\begin{aligned} x_p(t) &= \mathrm{Re}(z(t)) \\ &= [4\cos 2t + 32\sin 2t]/65. \end{aligned}$$

# Polynomial Forcing Term

$$y'' + py' + qy = P(t)$$

- Example: $y'' - 3y' + 2y = 1 - 4t.$
  - Try $y(t) = a + bt.$
  - Particular solution: $y(t) = -5 - 2t.$
- General solution

$$y(t) = -5 - 2t + C_1 e^t + C_2 e^{2t}.$$

# Exceptional Cases

- Example: $y'' - 3y' + 2y = 3e^t$.
  - Try $y(t) = ae^t$
  - The method does not work because $e^t$ is a solution to the associated homogeneous equation.
- Try $y(t) = ate^t$
  - Particular solution: $y_p(t) = -3te^t$.
- General solution: $y(t) = -3te^t + C_1e^t + C_2e^{2t}$.
- If the suggested particular solution does not work, multiply it by $t$ and try again.

# Combination Forcing Term

Example $y'' + 5y' + 6y = 2e^{2t} - 5\cos t$

- Solve

$$y_1'' + 5y_1' + 6y_1 = 2e^{2t}$$

$$y_2'' + 5y_2' + 6y_2 = -5\cos t$$

- Set $y(t) = y_1(t) + y_2(t)$.