

Math 211

Lecture #34

Inhomogeneous Equations
Forced Harmonic Motion

November 15, 2002

Inhomogeneous Equations

Theorem: Assume

- $y_p(t)$ is a particular solution to the inhomogeneous equation $y'' + py' + qy = f(t)$;
- $y_1(t)$ & $y_2(t)$ is a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$.

Then the general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + C_1y_1(t) + C_2y_2(t).$$

Method of Undetermined Coefficients

$$y'' + py' + qy = f(t)$$

The mantra for finding a particular solution is as follows:

- If the forcing term $f(t)$ has a form which is replicated under differentiation, then look for a particular solution of the same general form as the forcing term.

Exponential Forcing Term

$$y'' + py' + qy = Ce^{bt}$$

- **Example:** $y'' + 3y' + 2y = 4e^{-3t}$
- Try $y_p(t) = ae^{-3t}$; a to be determined.
 - ◆ Particular solution: $y_p(t) = 2e^{-3t}$.
- **Homogeneous equation:** $y'' + 3y' + 2y = 0$.
 - ◆ Fundamental set of solutions: e^{-2t} & e^{-t} .
- General solution to the inhomogeneous equation:

$$y(t) = 2e^{-3t} + C_1e^{-t} + C_2e^{-2t}.$$

Trigonometric Forcing Term

$$y'' + py' + qy = A \cos \omega t + B \sin \omega t$$

- **Example:** $y'' + 4y' + 5y = 4 \cos 2t - 3 \sin 2t$
- Try $y_p(t) = a \cos 2t + b \sin 2t$
 - ◆ Particular solution: $y_p(t) = [28 \cos 2t + 29 \sin 2t]/65$.
- **Homogeneous equation:** $y'' + 4y' + 5y = 0$
 - ◆ Fund. set of sol'ns: $e^{-2t} \cos t$ & $e^{-2t} \sin t$.
- General solution to the inhomogeneous equation:

$$y(t) = \frac{28 \cos 2t + 29 \sin 2t}{65} + e^{-2t} [C_1 \cos t + C_2 \sin t].$$

Complex Method

$$x'' + px' + qx = A \cos \omega t \quad \text{or}$$

$$y'' + py' + qy = A \sin \omega t.$$

- Solve $z'' + pz' + qz = Ae^{i\omega t}$.
 - ◆ Try $z(t) = ae^{i\omega t}$.

- Then

$$x_p(t) = \operatorname{Re}(z(t)) \quad \text{and}$$

$$y_p(t) = \operatorname{Im}(z(t)).$$

Example

$$x'' + 4x' + 5x = 4 \cos 2t$$

- Solve $z'' + 4z' + 5z = 4e^{2it}$.
 - ◆ Try $z(t) = ae^{2it}$.
 - ◆ Particular solution: $z(t) = (4 - 32i)e^{2it}/65$.
- Particular solution to the real equation:

$$\begin{aligned}x_p(t) &= \operatorname{Re}(z(t)) \\ &= [4 \cos 2t + 32 \sin 2t] / 65.\end{aligned}$$

Polynomial Forcing Term

$$y'' + py' + qy = P(t)$$

- Example: $y'' - 3y' + 2y = 1 - 4t$.
 - ◆ Try $y(t) = a + bt$.
 - ◆ Particular solution: $y(t) = -5 - 2t$.
- General solution

$$y(t) = -5 - 2t + C_1e^t + C_2e^{2t}.$$

Exceptional Cases

- Example: $y'' - 3y' + 2y = 3e^t$.
 - ◆ Try $y(t) = ae^t$
 - ◆ The method does not work because e^t is a solution to the associated homogeneous equation.
- Try $y(t) = ate^t$
 - ◆ Particular solution: $y_p(t) = -3te^t$.
- General solution: $y(t) = -3te^t + C_1e^t + C_2e^{2t}$.
- If the suggested particular solution does not work, multiply it by t and try again.

Combination Forcing Term

Example $y'' + 5y' + 6y = 2e^{2t} - 5 \cos t$

- Solve

$$y_1'' + 5y_1' + 6y_1 = 2e^{2t}$$

$$y_2'' + 5y_2' + 6y_2 = -5 \cos t$$

- Set $y(t) = y_1(t) + y_2(t)$.

Forced Harmonic Motion

Assume an oscillatory forcing term:

$$y'' + 2cy' + \omega_0^2 y = A \cos \omega t$$

- A is the forcing amplitude
- ω is the forcing frequency
- ω_0 is the natural frequency.
- c is the damping constant.

Forced Undamped Motion

$$y'' + \omega_0^2 y = A \cos \omega t$$

- Homogeneous equation

$$y'' + \omega_0^2 y = 0$$

- ♦ General solution

$$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

- ♦ If $\omega = \omega_0$ we have an exceptional case.

- $\omega \neq \omega_0$

$$y'' + \omega_0^2 y = A \cos \omega t$$

- ♦ Look for a particular solution of the form

$$x_p(t) = a \cos \omega t + b \sin \omega t.$$

- ♦ We find

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

- $\omega \neq \omega_0$

- ◆ General solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

- ◆ Initial conditions $x(0) = x'(0) = 0 \Rightarrow$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2} [\cos \omega t - \cos \omega_0 t].$$

- ◆ Example: $\omega_0 = 9, \omega = 8, A = \omega_0^2 - \omega^2 = 17.$

$$x(t) = \cos 9t - \cos 8t.$$

- $\omega \neq \omega_0$

- ♦ Set $\bar{\omega} = \frac{\omega_0 + \omega}{2}$ and $\delta = \frac{\omega_0 - \omega}{2}$.

$$\Rightarrow \omega = \bar{\omega} - \delta \quad \text{and} \quad \omega_0 = \bar{\omega} + \delta, \quad \text{and}$$

$$\begin{aligned} x(t) &= \frac{A}{\omega_0^2 - \omega^2} [\cos \omega t - \cos \omega_0 t] \\ &= \frac{A \sin \delta t}{2\bar{\omega}\delta} \sin \bar{\omega} t. \end{aligned}$$

- $\omega \neq \omega_0$

- ♦ Example:

$$\bar{\omega} = 8.5 \quad \text{and} \quad \delta = 0.5.$$

- ♦ Envelope: Slow oscillation with frequency δ ,

$$\pm \left| \frac{A \sin \delta t}{2\bar{\omega}\delta} \right|.$$

- ♦ Fast oscillation with frequency $\bar{\omega}$ and varying amplitude.
- ♦ Beats.

- $\omega = \omega_0$

$$y'' + \omega_0^2 y = A \cos \omega_0 t.$$

- ◆ We have an **exceptional case**. Try

$$x_p(t) = t[a \cos \omega t + b \sin \omega t].$$

- ◆ We find

$$x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t.$$

- ◆ General solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{2\omega_0} t \sin \omega_0 t.$$

- $\omega = \omega_0$

- ♦ **Initial conditions** $x(0) = x'(0) = 0 \Rightarrow$

$$x(t) = \frac{A}{2\omega_0} t \sin \omega_0 t.$$

- ▶ Example: $\omega_0 = 5$, and $A = 2\omega_0 = 10$.

$$x(t) = t \sin 5t.$$

- ♦ Oscillation with increasing amplitude.
- ♦ First example of *resonance*.
 - ▶ Forcing at the natural frequency can cause oscillations that grow out of control.