

# Math 211

Lecture #35

Forced Harmonic Motion

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# Forced Harmonic Motion

Assume an oscillatory forcing term:

$$y'' + 2cy' + \omega_0^2 y = A \cos \omega t$$

- $A$  is the forcing amplitude
- $\omega$  is the forcing frequency
- $\omega_0$  is the natural frequency.
- $c$  is the damping constant.

## Forced Undamped Motion

$$y'' + \omega_0^2 y = A \cos \omega t, \quad \text{where } \omega \neq \omega_0$$

- The solution with initial conditions  $x(0) = x'(0) = 0$ :

$$x(t) = \frac{A}{\omega_0^2 - \omega^2} [\cos \omega t - \cos \omega_0 t] = \frac{A \sin \delta t}{2\bar{\omega}\delta} \sin \bar{\omega} t,$$

$$\text{where } \bar{\omega} = \frac{\omega_0 + \omega}{2} \quad \text{and} \quad \delta = \frac{\omega_0 - \omega}{2}.$$

- This is a fast oscillation at frequency  $\bar{\omega}$ , with amplitude oscillating slowly with frequency  $\delta$ .
  - ♦ This phenomenon is called “beats.”

## Forced Undamped Motion (cont.)

$$y'' + \omega_0^2 y = A \cos \omega_0 t, \quad \text{where } \omega = \omega_0$$

- An exceptional case.
- Solution with initial conditions  $x(0) = x'(0) = 0$ :

$$x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t.$$

- The output is an oscillation with increasing amplitude.
- First example of *resonance*.
  - ♦ Forcing at the natural frequency can cause oscillations that grow out of control.

## Forced, Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

Use the complex method.

- Solve  $z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$ .
- We try  $z(t) = ae^{i\omega t}$  and get

$$\begin{aligned} z'' + 2cz' + \omega_0^2 z &= [(i\omega)^2 + 2c(i\omega) + \omega_0^2]ae^{i\omega t} \\ &= P(i\omega)z \end{aligned} ,$$

where  $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$  is the characteristic polynomial.

- The complex solution is  $z(t) = \frac{1}{P(i\omega)} Ae^{i\omega t}$ .
- The real solution is  $x_p(t) = \operatorname{Re}(z(t))$ .

## Example

$$x'' + 5x' + 4x = 50 \cos 3t$$

- $P(\lambda) = \lambda^2 + 5\lambda + 4.$ 
  - ♦  $P(i\omega) = P(3i) = -5 + 15i$
- $z(t) = \frac{1}{P(i\omega)} \cdot 50e^{3it}$ 
$$= -[(\cos 3t - 3 \sin 3t) + i(\sin 3t + 3 \cos 3t)]$$
- $x_p(t) = \operatorname{Re}(z(t)) = 3 \sin 3t - \cos 3t.$

# The Transfer Function

- The **complex solution** is

$$z(t) = \frac{1}{P(i\omega)} A e^{i\omega t} = H(i\omega) A e^{i\omega t},$$

where  $H(i\omega) = \frac{1}{P(i\omega)}$  is called the **transfer function**.

- We will use complex polar coordinates to write

$$H(i\omega) = G(\omega) e^{-i\phi(\omega)},$$

where  $G(\omega) = |H(i\omega)|$  is called the **gain** and  $\phi(\omega)$  is called the **phase shift**.

## The Gain and Phase Shift

- If  $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$  is the characteristic polynomial, then  $P(i\omega) = Re^{i\phi}$ , where

$$R = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}, \quad \text{and}$$

$$\phi = \operatorname{arccot} \left( \frac{\omega_0^2 - \omega^2}{2c\omega} \right).$$

- The transfer function is

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R}e^{-i\phi} = G(\omega)e^{-i\phi}.$$

- ♦ The gain  $G(\omega) = \frac{1}{R} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$ .

- The **complex particular solution** is

$$\begin{aligned}z(t) &= H(i\omega)Ae^{i\omega t} = G(\omega)e^{-i\phi} \cdot Ae^{i\omega t} \\ &= G(\omega)Ae^{i(\omega t - \phi)}.\end{aligned}$$

- The real particular solution is

$$x_p(t) = \operatorname{Re}(z(t)) = G(\omega)A \cos(\omega t - \phi).$$

- ♦ The amplitude of  $x_p$  is  $G(\omega)A$ , and the phase is  $\phi$ .

- The general solution is

$$\begin{aligned}x(t) &= x_p(t) + x_h(t) \\ &= G(\omega)A \cos(\omega t - \phi) + x_h(t),\end{aligned}$$

where  $x_h(t)$  is the general solution of the homogeneous equation.

- $x_h(t) \rightarrow 0$  as  $t$  increases, so  $x_h$  is called the *transient term*.
- $x_p(t) = G(\omega)A \cos(\omega t - \phi)$  is called the *steady-state solution*.

# Example

$$x'' + 5x' + 4x = 50 \cos 3t$$

- $G(\omega) = \frac{1}{\sqrt{(4 - \omega^2)^2 + 25\omega^2}}$  and

$$\phi = \operatorname{arccot} \left( \frac{4 - \omega^2}{5\omega} \right).$$

- ♦ With  $\omega = 3$ ,

$$G(3) = \frac{1}{5\sqrt{10}} \approx 0.0632$$

$$\phi = \operatorname{arccot}(-3/5) \approx 2.1112.$$

- ♦ **SS solution**  $x_p(t) = G(3)A \cos(3t - \phi)$ .

# The Steady-State Solution

$$x_p(t) = G(\omega)A \cos(\omega t - \phi).$$

- The **forcing function** is  $A \cos \omega t$ .
- Properties of the steady-state response:
  - ◆ It is oscillatory at the driving frequency.
  - ◆ The amplitude is the product of the gain,  $G(\omega)$ , and the amplitude of the forcing function.
  - ◆ It has a phase shift of  $\phi$  with respect to the forcing function.

## The Gain

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$$

Set  $\omega = s\omega_0$  and  $c = D\omega_0/2$  (or  $s = \omega/\omega_0$  and  $D = 2c/\omega_0$ ). Then

$$G(\omega) = \frac{1}{\omega_0^2} \frac{1}{\sqrt{(1 - s^2)^2 + D^2 s^2}}$$