

Math 211

Lecture #36

Nonlinear Systems

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Interacting Species

- Two species with populations x_1 & x_2 .
- Interaction between the species can be helpful or detrimental.
- Basic model

$$x_1' = r_1 x_1$$

$$x_2' = r_2 x_2$$

- ♦ r_1 & r_2 are the *reproductive rates*.

Reproductive Rates

- If $x_2 = 0$ the reproductive rate for x_1 is

$$r_1 = a_1 - b_1 x_1.$$

- ♦ $a_1 > 0 \Rightarrow$ natural growth.
- ♦ $a_1 < 0 \Rightarrow$ natural decline.
- ♦ $b_1 = 0$ Malthusian growth.
- ♦ $b_1 > 0$ logistic growth.

- If $x_2 > 0$ the **reproductive rate** for x_1 is

$$r_1 = a_1 - b_1x_1 + c_1x_2.$$

- ♦ $c_1 > 0 \Rightarrow$ interaction is helpful to x_1 .
- ♦ $c_1 < 0 \Rightarrow$ interaction is detrimental to x_1 .

- The reproductive rate for x_2 is

$$r_2 = a_2 - b_2x_2 + c_2x_1.$$

- The model for **interacting species** is

$$x'_1 = (a_1 - b_1x_1 + c_1x_2)x_1$$

$$x'_2 = (a_2 - b_2x_2 + c_2x_1)x_2$$

Predator Prey Model

Rabbits & foxes, fish & sharks, and cottony cushion scale insect & ladybird beetle.

- F = fish & S = sharks.

$$F' = (a - bS)F$$

$$S' = (-c + dF)S$$

or

$$F' = (a - eF - bS)F$$

$$S' = (-c + dF)S$$

$$a = 3, b = 3, c = 1, d = 3, e = 3.$$

Competing Species

Cattle and sheep.

- x_1 and x_2 competing for resources.

$$x'_1 = (a_1 - b_1x_1 + c_1x_2)x_1$$

$$x'_2 = (a_2 - b_2x_2 + c_2x_1)x_2$$

- ♦ $a_i > 0$, $b_i > 0$, & $c_i < 0$

- Example:

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

Linearization

The principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.
- Example: Approximate the function $f(y)$ near y_0 by a linear function.

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

$$\text{where } \lim_{h \rightarrow 0} \frac{R(h)}{h} = 0.$$

- ♦ The linear function is $L(h) = f(y_0) + f'(y_0)h$.

Linearization of an ODE

$$y' = f(y)$$

- Assume $f(y_0) = 0$ and $f'(y_0) \neq 0$.
- Set $y = y_0 + u$. Get

$$\begin{aligned}u' &= f(y_0 + u) \\ &= f'(y_0)u + R(u)\end{aligned}$$

- Approximate by the linear differential equation

$$\tilde{u}' = f'(y_0)\tilde{u}$$

- If $f'(y_0) \neq 0$ the equilibrium point of the **linearization** at 0 has the same stability properties as that of the nonlinear equation at y_0 .
 - ◆ $f'(y_0) > 0 \Rightarrow y_0$ is unstable.
 - ◆ $f'(y_0) < 0 \Rightarrow y_0$ is asymptotically stable.
- We can solve the linearization explicitly.

Linearization of a Planar System

$$x' = f(x, y)$$

$$y' = g(x, y)$$

- Assume (x_0, y_0) is an equilibrium point, so

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

We have by Taylor's theorem

$$\begin{aligned} f(x_0 + u, y_0 + v) \\ &= \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v) \end{aligned}$$

$$\begin{aligned} g(x_0 + u, y_0 + v) \\ &= \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v) \end{aligned}$$

$$\text{where } \frac{R_f(u, v)}{\sqrt{u^2 + v^2}} \rightarrow 0 \text{ and } \frac{R_g(u, v)}{\sqrt{u^2 + v^2}} \rightarrow 0$$

- Set $x = x_0 + u$ and $y = y_0 + v$. The **system** becomes

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$

$$v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

Linearization at (x_0, y_0)

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0)\tilde{v}$$

$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0)\tilde{v}$$

- This is a linear system.
 - ◆ We can solve it explicitly.
 - ◆ Does it give information about the original nonlinear system?

Matrix Form of the Linearization

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

Theorem: Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the **linearization** at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
 - ◆ All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
 - ◆ Center and others. Occupy pieces of the boundaries.

Examples

- Predator prey
- Competing species
- Center

$$x' = y + \alpha x(x^2 + y^2)$$

$$y' = -x + \alpha y(x^2 + y^2)$$