Math 211

Lecture #37

The Use of the Linearization

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Linearization

The principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.
- Example: Approximate the function $f(y)$ near y_0 by a linear function.

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

$$\text{where } \lim_{h \to 0} \frac{R(h)}{h} = 0.$$

  - ◆ The linear function is $L(h) = f(y_0) + f'(y_0)h$.

Linearization of a Planar System

$$x' = f(x, y)$$

$$y' = g(x, y)$$

- (x_0, y_0) is an equilibrium point so

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

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Taylor's Theorem

- We have by Taylor's theorem

$$
\begin{aligned}
& f(x_0 + u, y_0 + v) \\
& \quad = \frac{\partial f}{\partial x}(x_0, y_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v + R_f(u, v) \\
& g(x_0 + u, y_0 + v) \\
& \quad = \frac{\partial g}{\partial x}(x_0, y_0) \cdot u + \frac{\partial g}{\partial y}(x_0, y_0) \cdot v + R_g(u, v)
\end{aligned}
$$

where $\dfrac{R_f(u, v)}{\sqrt{u^2 + v^2}} \to 0$ and $\dfrac{R_g(u, v)}{\sqrt{u^2 + v^2}} \to 0$

The Linearization at (x_0, y_0)

- Set $x = x_0 + u$ and $y = y_0 + v$. The system becomes

$$\begin{aligned} u' &= \frac{\partial f}{\partial x}(x_0, y_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v + R_f(u, v) \\ v' &= \frac{\partial g}{\partial x}(x_0, y_0) \cdot u + \frac{\partial g}{\partial y}(x_0, y_0) \cdot v + R_g(u, v) \end{aligned}$$

- The linearization is defined to be

$$\begin{aligned} \tilde{u}' &= \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v} \\ \tilde{v}' &= \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v} \end{aligned}$$

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Matrix Form of the Linearization

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0, y_0) & \dfrac{\partial f}{\partial y}(x_0, y_0) \\ \dfrac{\partial g}{\partial x}(x_0, y_0) & \dfrac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

- The behavior of solutions to the linearization is determined by the eigenvalues of the Jacobian.

Properties of the Linearization

- The linearization gives us information about the original system.

Theorem: Consider the planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
    - ♦ All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
    - ♦ Center and others, which occupy pieces of the boundaries between the generic points.

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Theorem

Examples

- Center.

$$x' = y + \alpha x(x^2 + y^2)$$

$$y' = -x + \alpha y(x^2 + y^2)$$

  - $\alpha > 0 \;\Rightarrow\; (0,0)^T$ is unstable.
  - $\alpha < 0 \;\Rightarrow\; (0,0)^T$ is a sink.
- Competing species.
- Default system in pplane.

Linear Analysis of Equilibrium Points

- Provides a good qualitative picture of how solutions behave near generic equilibrium points.
- Provides limited qualitative information about the solutions near nongeneric equilibrium points.
    - ♦ A linear center could be a spiral source or a spiral sink.
- Provides no information about the global behavior of solutions to nonlinear systems.

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Higher Dimensional Systems

Autonomous equation $\mathbf{y}' = \mathbf{f}(\mathbf{y})$.

- $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$, $\mathbf{y}_0$ is an equilibrium point.
- $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \cdots, f_n(\mathbf{y}))^T$
- J is the Jacobian matrix
- $\mathbf{f}(\mathbf{y}_0 + \mathbf{u}) = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u})$ where $\lim_{\mathbf{u}\to\mathbf{0}} \frac{\mathbf{R}(\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}$.
- Set $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$. The system becomes

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}).$$

- The linearization is $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$.

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Theorem: Suppose that $\mathbf{y}_0$ is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of $\mathbf{f}$ at $\mathbf{y}_0$.

1. Suppose that the real part of every eigenvalue of J is negative. Then $\mathbf{y}_0$ is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then $\mathbf{y}_0$ is an unstable equilibrium point.

Example

$$x' = -2x - 4y + 2xy$$

$$y' = x - 6y + x^2 - y^2$$

- The origin $(0, 0)$ is an equilibrium point.
- The Jacobian has one eigenvalue, $\lambda = -4$, of algebraic multiplicity 2.
- First theorem does not apply.
- Second theorem $\Rightarrow$ the origin is a sink.