Math 211

Lecture #38

The Linearization in Higher Dimension

November 25, 2002

Higher Dimensional Systems

Autonomous equation $\mathbf{y}' = \mathbf{f}(\mathbf{y})$.

- $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$, $\mathbf{y}_0$ is an equilibrium point.
- $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \cdots, f_n(\mathbf{y}))^T$
- J is the Jacobian matrix
- $\mathbf{f}(\mathbf{y}_0 + \mathbf{u}) = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u})$ where $\lim_{\mathbf{u}\to\mathbf{0}} \frac{\mathbf{R}(\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}$.
- Set $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$. The system becomes

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}).$$

- The linearization is $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$.

Return

Linearization of a Planar System

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad f(x_0,y_0) = g(x_0,y_0) = 0.$$

- The linearization at (x_0, y_0) is

$$\begin{aligned} \tilde{u}' &= \frac{\partial f}{\partial x}(x_0,y_0)\cdot\tilde{u} + \frac{\partial f}{\partial y}(x_0,y_0)\cdot\tilde{v} \\ \tilde{v}' &= \frac{\partial g}{\partial x}(x_0,y_0)\cdot\tilde{u} + \frac{\partial g}{\partial y}(x_0,y_0)\cdot\tilde{v} \end{aligned}$$

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The Jacobian

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$. The *Jacobian matrix* is

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

Theorem: Consider the planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Theorem: Suppose that $\mathbf{y}_0$ is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of $\mathbf{f}$ at $\mathbf{y}_0$.

1. Suppose that the real part of every eigenvalue of J is negative. Then $\mathbf{y}_0$ is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then $\mathbf{y}_0$ is an unstable equilibrium point.

Example

$$x' = -2x - 4y + 2xy$$
$$y' = x - 6y + x^2 - y^2$$

- The origin $(0,0)$ is an equilibrium point.
- The Jacobian has one eigenvalue, $\lambda = -4$, of algebraic multiplicity 2.
- First theorem does not apply.
- Second theorem $\Rightarrow$ the origin is a sink.

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The Lorenz System

$$x' = -ax + ay$$
$$y' = rx - y - xz$$
$$z' = -bz + xy$$

- Equilibrium points.
  - $(r \le 1)$ $(0,0,0)$
  - $(r > 1)$ Set $s = \sqrt{b(r-1)}$. The equilibrium points are $(0,0,0)$, and $\mathbf{c}^{\pm} = (\pm s, \pm s, r-1)$.

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- The Jacobian is

$$J = \begin{pmatrix} -a & a & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

  - Use $a = 10$ and $b = 8/3$.
  - $(0,0,0)$
    - If $r < 1$ $(0,0,0)$ is asymptotically stable.
    - If $r > 1$ $(0,0,0)$ is unstable.

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- $\mathbf{c}^+$ and $\mathbf{c}^-$
  - For $1 < r < 470/19 \approx 24.74$, $\mathbf{c}^+$ and $\mathbf{c}^-$ are asymptotically stable.
  - For $r > 470/19 \approx 24.74$, $\mathbf{c}^+$ and $\mathbf{c}^-$ are unstable.
- As r varies the Lorenz system displays a wide variety of behaviors.
  - For $r = 28$ we have Lorenz's strange attractor.
  - For $r = 100$ there is a periodic attractor.
  - For $r = 200$ there is another strange attractor.

Return Jacobian $\mathbf{c}^+$&$\mathbf{c}^-$

Invariant Sets

Definition: A set S is *(positively) invariant* for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples:
  - An equilibrium point.
  - Any solution curve.

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Example — Competing Species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- The positive x- and y-axes are invariant.
- The positive quadrant is invariant.
  - Populations should remain nonnegative.
- The set $S = \{(x, y) \,|\, 0 < x < 3,\ 0 < y < 3\}$ is positively invariant.

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Nullclines

$$x' = f(x, y)$$
$$y' = g(x, y)$$

Definition: The x-*nullcline* is the set defined by $f(x, y) = 0$. The y-*nullcline* is the set defined by $g(x, y) = 0$.

- Along the x-nullcline the vector field points up or down.
- Along the y-nullcline the vector field points left or right.
- The nullclines intersect at the equilibrium points.

Competing Species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- x-nullcline: two lines $x = 0$ and $2x + y = 5$.
- y-nullcline: two lines $y = 0$ and $2x + 3y = 7$.
- Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant $\rightarrow (2, 1)$ as $t \rightarrow \infty$.