

# Math 211

Lecture #38

The Linearization in Higher Dimension

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# Higher Dimensional Systems

Autonomous equation  $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ .

- $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ ,  $\mathbf{y}_0$  is an equilibrium point.
- $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \dots, f_n(\mathbf{y}))^T$
- $J$  is the **Jacobian matrix**
- $\mathbf{f}(\mathbf{y}_0 + \mathbf{u}) = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u})$  where  $\lim_{\mathbf{u} \rightarrow \mathbf{0}} \frac{\mathbf{R}(\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}$ .
- Set  $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$ . The system becomes

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}).$$

- The linearization is  $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$ .

## Linearization of a Planar System

$$\begin{aligned}x' &= f(x, y) \\y' &= g(x, y)\end{aligned}\quad f(x_0, y_0) = g(x_0, y_0) = 0.$$

- The linearization at  $(x_0, y_0)$  is

$$\begin{aligned}\tilde{u}' &= \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v} \\ \tilde{v}' &= \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v}\end{aligned}$$

# The Jacobian

Set  $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ . The *Jacobian matrix* is

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

**Theorem:** Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where  $f$  and  $g$  are continuously differentiable. Suppose that  $(x_0, y_0)$  is an equilibrium point. If the **linearization** at  $(x_0, y_0)$  has a generic equilibrium point at the origin, then the equilibrium point at  $(x_0, y_0)$  is of the same type.

**Theorem:** Suppose that  $\mathbf{y}_0$  is an equilibrium point for  $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ . Let  $J$  be the Jacobian of  $\mathbf{f}$  at  $\mathbf{y}_0$ .

1. Suppose that the real part of every eigenvalue of  $J$  is negative. Then  $\mathbf{y}_0$  is an asymptotically stable equilibrium point.
2. Suppose that  $J$  has at least one eigenvalue with positive real part. Then  $\mathbf{y}_0$  is an unstable equilibrium point.

## Example

$$x' = -2x - 4y + 2xy$$

$$y' = x - 6y + x^2 - y^2$$

- The origin  $(0, 0)$  is an equilibrium point.
- The Jacobian has one eigenvalue,  $\lambda = -4$ , of algebraic multiplicity 2.
- **First theorem** does not apply.
- **Second theorem**  $\Rightarrow$  the origin is a sink.

# The Lorenz System

$$x' = -ax + ay$$

$$y' = rx - y - xz$$

$$z' = -bz + xy$$

- Equilibrium points.
  - ◆  $(r \leq 1)$   $(0, 0, 0)$
  - ◆  $(r > 1)$  Set  $s = \sqrt{b(r - 1)}$ . The equilibrium points are  $(0, 0, 0)$ , and  $\mathbf{c}^{\pm} = (\pm s, \pm s, r - 1)$ .

- The Jacobian is

$$J = \begin{pmatrix} -a & a & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

- ♦ Use  $a = 10$  and  $b = 8/3$ .
- ♦  $(0, 0, 0)$ 
  - ▶ If  $r < 1$   $(0, 0, 0)$  is asymptotically stable.
  - ▶ If  $r > 1$   $(0, 0, 0)$  is unstable.

- ◆  $c^+$  and  $c^-$ 
  - ▶ For  $1 < r < 470/19 \approx 24.74$ ,  $c^+$  and  $c^-$  are asymptotically stable.
  - ▶ For  $r > 470/19 \approx 24.74$ ,  $c^+$  and  $c^-$  are unstable.
- As  $r$  varies the **Lorenz system** displays a wide variety of behaviors.
  - ◆ For  $r = 28$  we have Lorenz's strange attractor.
  - ◆ For  $r = 100$  there is a periodic attractor.
  - ◆ For  $r = 200$  there is another strange attractor.

# Invariant Sets

**Definition:** A set  $S$  is *(positively) invariant* for the system  $\mathbf{y}' = \mathbf{f}(\mathbf{y})$  if  $\mathbf{y}(0) = \mathbf{y}_0 \in S$  implies that  $\mathbf{y}(t) \in S$  for all  $t \geq 0$ .

- Examples:
  - ◆ An equilibrium point.
  - ◆ Any solution curve.

## Example — Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

- The positive  $x$ - and  $y$ -axes are **invariant**.
- The positive quadrant is invariant.
  - ♦ Populations should remain nonnegative.
- The set  $S = \{(x, y) \mid 0 < x < 3, 0 < y < 3\}$  is positively invariant.

## Nullclines

$$x' = f(x, y)$$

$$y' = g(x, y)$$

**Definition:** The *x-nullcline* is the set defined by  $f(x, y) = 0$ . The *y-nullcline* is the set defined by  $g(x, y) = 0$ .

- Along the *x*-nullcline the vector field points up or down.
- Along the *y*-nullcline the vector field points left or right.
- The nullclines intersect at the equilibrium points.

# Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

- $x$ -nullcline: two lines  $x = 0$  and  $2x + y = 5$ .
- $y$ -nullcline: two lines  $y = 0$  and  $2x + 3y = 7$ .
- Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant  $\rightarrow (2, 1)$  as  $t \rightarrow \infty$ .