

# Math 211

Lecture #41

Limit Cycles and the Pendulum

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## Basic Question about $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- The (forward) limit set of the solution  $\mathbf{y}(t)$  that starts at  $\mathbf{y}_0$  is the set of all limit points of the solution curve. It is denoted by  $\omega(\mathbf{y}_0)$ .
  - ♦  $\mathbf{x} \in \omega(\mathbf{y}_0)$  if there is a sequence  $t_k \rightarrow \infty$  such that  $\mathbf{y}(t_k) \rightarrow \mathbf{x}$ .
- What is  $\omega(\mathbf{y}_0)$  for all  $\mathbf{y}_0$ ?

## Properties of Limit Sets

**Theorem:** Suppose that the system  $\mathbf{y}' = \mathbf{f}(\mathbf{y})$  is defined in the set  $U$ .

1. If the solution curve starting at  $\mathbf{y}_0$  stays in a bounded subset of  $U$ , then the limit set  $\omega(\mathbf{y}_0)$  is not empty.
2. Any limit set is both positively and negatively invariant.

## Example

$$x' = 5y + x(9 - x^2 - y^2)$$

$$y' = -5x + y(9 - x^2 - y^2)$$

- The origin is a spiral source.
- In polar coordinates the system is

$$r' = r(9 - r^2)$$

$$\theta' = -5$$

- All solution curves approach the circle  $x^2 + y^2 = 9$ .
  - ♦ The circle  $x^2 + y^2 = 9$  is a solution curve.

## Limit Cycle

**Definition:** A **limit cycle** is a closed solution curve which is the limit set of nearby solution curves. If the solution curves spiral into the limit cycle as  $t \rightarrow \infty$ , it is a **attracting limit cycle**. If they spiral into the limit cycle as  $t \rightarrow -\infty$ , it is a **repelling limit cycle**.

- In the **example** the circle  $x^2 + y^2 = 9$  is an attracting limit cycle.

## Types of Limit Set

- A limit cycle is a new type of phenomenon. However, the limit set is a periodic orbit, so the type of limit set is not new.
- We still have only two **types** of non-empty limits sets.
  - ◆ An equilibrium point.
  - ◆ A closed solution curve.

## Example

$$x' = (x + 1)(x + 2y)(1 - (x + y - 1)/5)$$

$$y' = -(y + 1)(2x + y)$$

- The lines  $x = -1$  and  $y = -1$  are invariant. The line  $x + y = 1$  is invariant. The triangle is invariant.
- The vertices of the triangle are saddle points. The sides are separatrices.
- The origin is a spiral source.
- The limit set of any solution that starts in the triangle is the boundary of the triangle. This is a new **type**.

# Planar Graph

**Definition:** A *planar graph* is a collection of points, called *vertices*, and non-intersecting curves, called *edges*, which connect the vertices. If the edges each have a direction the graph is said to be *directed*.

- The boundary of the triangle in the *example* is a directed planar graph.
- Look at Exercises 14 – 22 in Section 10.4.

**Theorem:** If  $S$  is a nonempty limit set of a solution of a planar system defined in a set  $U \subset \mathbf{R}^2$ , then  $S$  is one of the following:

- An equilibrium point.
- A closed solution curve.
- A **directed planar graph** with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives*.

## Poincaré-Bendixson Theorem

**Theorem:** Suppose that  $R$  is a closed and bounded planar region that is positively invariant for a planar system. If  $R$  contains no equilibrium points, then there is a closed solution curve in  $R$ .

- The theorem is also true if the set  $R$  is negatively invariant.
- The closed solution curve might be a limit cycle.

# Examples

- # 1.

$$x' = x + y - x(x^2 + 3y^2)$$

$$y' = -x + y - 2y^3$$

- ◆ The set  $\{(x, y) \mid 0.5 \leq x^2 + y^2 \leq 1\}$  is positively invariant.
- # 2. Rayleigh's example:  $z'' + \mu z'[(z')^2 - 1] + z = 0$ .
  - ◆ There is a limit cycle.

# The Pendulum

- The angle  $\theta$  satisfies the nonlinear differential equation

$$mL\theta'' = -mg \sin \theta - D \theta',$$

or

$$\theta'' + \frac{D}{mL}\theta' + \frac{g}{L} \sin \theta = 0.$$

- ♦ We will write this as

$$\theta'' + d\theta + b \sin \theta = 0.$$

# The Pendulum System

- Introduce  $\omega = \theta'$  to get the system

$$\theta' = \omega$$

$$\omega' = -b \sin \theta - d\omega$$

- The equilibrium points are  $(k\pi, 0)^T$  where  $k$  is any integer.
  - ♦ If  $k$  is odd the equilibrium point is a saddle.
  - ♦ If  $k$  is even the equilibrium point is a center if  $d = 0$  or a sink if  $d > 0$ .