

## Math 211

Lecture #42

The Pendulum

Predator-Prey

December 6, 2002

### The Pendulum

- The angle  $\theta$  satisfies the nonlinear differential equation

$$mL\theta'' = -mg \sin \theta - D\theta',$$

- ♦ We will write this as

$$\theta'' + d\theta + b \sin \theta = 0.$$

- Introduce  $\omega = \theta'$  to get the system

$$\theta' = \omega$$

$$\omega' = -b \sin \theta - d\omega$$

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### Analysis

- The equilibrium points are  $(k\pi, 0)^T$  where  $k$  is any integer.
  - ♦ If  $k$  is odd the equilibrium point is a saddle.
  - ♦ If  $k$  is even the equilibrium point is a center if  $d = 0$  or a sink if  $d > 0$ .

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## The Inverted Pendulum

- The angle  $\theta$  measured from straight up satisfies the nonlinear differential equation

$$mL\theta'' = mg \sin \theta - D\theta',$$

or

$$\theta'' + \frac{D}{mL}\theta' - \frac{g}{L}\sin \theta = 0.$$

- We will write this as

$$\theta'' + d\theta - b \sin \theta = 0.$$

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## The Inverted Pendulum System

- Introduce  $\omega = \theta'$  to get the system

$$\theta' = \omega$$

$$\omega' = b \sin \theta - d\omega$$

- The equilibrium point at  $(0, 0)^T$  is a saddle point and unstable.
- Can we find an automatic way of sensing the departure of the system from  $(0, 0)^T$  and moving the pivot to bring the system back to the unstable point at  $(0, 0)^T$ ?
  - Experimentally the answer is yes.

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## The Control System

- If we apply a force  $v$  moving the pivot to the right or left, then  $\theta$  satisfies

$$mL\theta'' = mg \sin \theta - D\theta' - v \cos \theta,$$

- The system becomes

$$\theta' = \omega$$

$$\omega' = b \sin \theta - d\omega - u \cos \theta,$$

where  $u = v/mL$ .

- Assume the force is a linear response to the detected value of  $\theta$ , so  $u = c\theta$ , where  $c$  is a constant.

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## The Controlled System

- The Jacobian at the origin is

$$J = \begin{pmatrix} 0 & 1 \\ b-c & -d \end{pmatrix}$$

- The origin is asymptotically stable if  $T = -d < 0$  and  $D = c - b > 0$ . Therefore require

$$c > b = \frac{g}{L}.$$

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## Predator-Prey

Lotka-Volterra system

$$x' = (a - by)x \quad (\text{prey - fish})$$

$$y' = (-c + dx)y \quad (\text{predator - sharks})$$

- Equilibrium points:  $(0, 0)$  is a saddle,  $(x_0, y_0) = (c/d, a/b)$  is a linear center.
- The axes are invariant.
- The positive quadrant is invariant.
- The solution curves appear to be closed. Is this actually true?

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## Solutions are Periodic

Along the solution curve  $y = y(x)$  we have

$$\frac{dy}{dx} = \frac{y(-c + dx)}{x(a - by)}.$$

The solution is

$$H(x, y) = by - a \ln y + dx - c \ln x = C$$

- This is an implicit equation for the solution curve.  $\Rightarrow$  All solution curves are closed, and represent periodic solutions.

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## Why Fishing Leads to More Fish

Compute the average of the fish & shark populations.

$$\frac{d}{dt} \ln x(t) = \frac{x'}{x} = a - by$$

$$0 = \frac{1}{T} \int_0^T \frac{d}{dt} \ln x(t) dt = a - b\bar{y}.$$

So  $\bar{y} = a/b = y_0$ . Similarly  $\bar{x} = x_0 = c/d$ .

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The effect of fishing that does not distinguish between fish and sharks is the system

$$x' = (a - by)x - ex$$

$$y' = (-c + dx)y - ey$$

This is the same system with  $a$  replaced by  $a - e$  and  $c$  replaced by  $c + e$ .

System

Averages

The average populations are

$$\bar{x}_1 = \frac{c+e}{d} \quad \text{and} \quad \bar{y}_1 = \frac{a-e}{b}$$

Fishing causes the average fish population to increase and the average shark population to decrease.

Averages

## Cottony Cushion Scale Insect & the Ladybird Beetle

- Cottony cushion scale insect accidentally introduced from Australia in 1868.
  - ♦ Threatened the citrus industry.
- Ladybird beetle imported from Australia
  - ♦ Natural predator – reduced the insects to manageable low.

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DDT kills the scale insect.

- Massive spraying ordered.
  - ♦ Despite the warnings of mathematicians and biologists.
- The scale insect increased in numbers, as predicted by Volterra.