

Math 211

Lecture #42

The Pendulum

Predator-Prey

December 6, 2002

The Pendulum

- The angle θ satisfies the nonlinear differential equation

$$mL\theta'' = -mg \sin \theta - D \theta',$$

- ♦ We will write this as

$$\theta'' + d\theta + b \sin \theta = 0.$$

- Introduce $\omega = \theta'$ to get the system

$$\theta' = \omega$$

$$\omega' = -b \sin \theta - d\omega$$

Analysis

- The equilibrium points are $(k\pi, 0)^T$ where k is any integer.
 - ◆ If k is odd the equilibrium point is a saddle.
 - ◆ If k is even the equilibrium point is a center if $d = 0$ or a sink if $d > 0$.

The Inverted Pendulum

- The angle θ measured from straight up satisfies the nonlinear differential equation

$$mL\theta'' = mg \sin \theta - D \theta',$$

or

$$\theta'' + \frac{D}{mL}\theta' - \frac{g}{L} \sin \theta = 0.$$

- ♦ We will write this as

$$\theta'' + d\theta - b \sin \theta = 0.$$

The Inverted Pendulum System

- Introduce $\omega = \theta'$ to get the system

$$\theta' = \omega$$

$$\omega' = b \sin \theta - d\omega$$

- The equilibrium point at $(0, 0)^T$ is a saddle point and unstable.
- Can we find an automatic way of sensing the departure of the system from $(0, 0)^T$ and moving the pivot to bring the system back to the unstable point at $(0, 0)^T$?
 - ◆ Experimentally the answer is yes.

The Control System

- If we apply a force v moving the pivot to the right or left, then θ satisfies

$$mL\theta'' = mg \sin \theta - D\theta' - v \cos \theta,$$

- The system becomes

$$\theta' = \omega$$

$$\omega' = b \sin \theta - d\omega - u \cos \theta,$$

where $u = v/mL$.

- Assume the force is a linear response to the detected value of θ , so $u = c\theta$, where c is a constant.

The Controlled System

- The Jacobian at the origin is

$$J = \begin{pmatrix} 0 & 1 \\ b - c & -d \end{pmatrix}$$

- The origin is asymptotically stable if $T = -d < 0$ and $D = c - b > 0$. Therefore require

$$c > b = \frac{g}{L}.$$

Predator-Prey

Lotka-Volterra system

$$x' = (a - by)x \quad (\text{prey} - \text{fish})$$

$$y' = (-c + dx)y \quad (\text{predator} - \text{sharks})$$

- Equilibrium points: $(0, 0)$ is a saddle, $(x_0, y_0) = (c/d, a/b)$ is a linear center.
- The axes are invariant.
- The positive quadrant is invariant.
- The solution curves appear to be closed. Is this actually true?

Solutions are Periodic

Along the solution curve $y = y(x)$ we have

$$\frac{dy}{dx} = \frac{y(-c + dx)}{x(a - by)}.$$

The solution is

$$H(x, y) = by - a \ln y + dx - c \ln x = C$$

- This is an implicit equation for the solution curve. \Rightarrow All solution curves are closed, and represent periodic solutions.

Why Fishing Leads to More Fish

Compute the average of the fish & shark populations.

$$\frac{d}{dt} \ln x(t) = \frac{x'}{x} = a - by$$

$$0 = \frac{1}{T} \int_0^T \frac{d}{dt} \ln x(t) dt = a - b\bar{y}.$$

So $\bar{y} = a/b = y_0$. Similarly $\bar{x} = x_0 = c/d$.

The effect of fishing that does not distinguish between fish and sharks is the system

$$x' = (a - by)x - ex$$

$$y' = (-c + dx)y - ey$$

This is the same system with a replaced by $a - e$ and c replaced by $c + e$.

The average populations are

$$\bar{x}_1 = \frac{c + e}{d} \quad \text{and} \quad \bar{y}_1 = \frac{a - e}{b}$$

Fishing causes the average fish population to increase and the average shark population to decrease.

Cottony Cushion Scale Insect & the Ladybird Beetle

- Cottony cushion scale insect accidentally introduced from Australia in 1868.
 - ◆ Threatened the citrus industry.
- Ladybird beetle imported from Australia
 - ◆ Natural predator – reduced the insects to manageable low.

DDT kills the scale insect.

- Massive spraying ordered.
 - ◆ Despite the warnings of mathematicians and biologists.
- The scale insect increased in numbers, as predicted by Volterra.