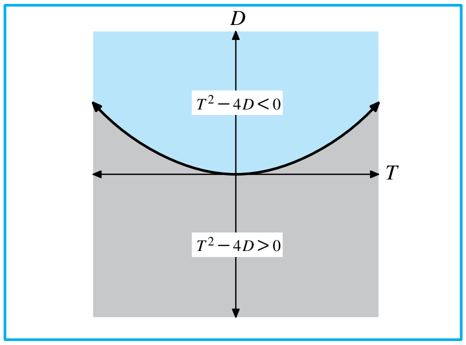
CHAPTER

9



Linear Systems with Constant Coefficients

We are now ready to turn to the task of finding the general solution to linear, homogeneous equations and systems. Remember the strategy we developed in the previous chapter. For a system of dimension n, we need to find a fundamental set of solutions, which is a set of n linearly independent solutions. The general solution is a linear combination of these solutions.

We will also find the general solution for higher-order equations. However, this will be an easy job, since we need only use the results for the associated first-order system.

9.1 Overview of the Technique

We are looking for a solution to the initial value problem

$$\mathbf{y}' = A\mathbf{y} \quad \text{with} \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{1.1}$$

where A is a matrix with constant entries. For motivation, let's look at a system of dimension 1, which we already understand. The initial value problem has the form

$$y' = ay \quad \text{with} \quad y(0) = y_0.$$

Using the techniques developed in Chapter 2, we find that the solution to this problem is the exponential function

$$y(t) = e^{at} y_0. \tag{1.2}$$

Is it possible that we can write down the solution to the general initial value problem in equation (1.1) by analogy to the solution for dimension 1 in (1.2)? This would mean that the solution is given by

$$\mathbf{y}(t) = e^{tA}\mathbf{y}_0. \tag{1.3}$$

At the moment, the term e^{tA}, the exponential of a matrix, makes no sense. Our task, therefore, is to make sense of the exponential of a matrix, to learn as much about it as we can, to show that (1.3) actually gives us a solution to the initial value problem in (1.1), and to learn how to compute it.

The exponential of a matrix and solutions to differential equations

The correct way to define the exponential of a matrix is not at all obvious. We will do so using a power series, in analogy to the power series for the exponential function,

$$e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}a^k. \tag{1.4}$$

DEFINITION 1.5 The ***exponential of the matrix*** A is defined to be

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k. \tag{1.6}$$

In this formula, $A^2 = AA$, $A^3 = AAA$, etc., all of the products being matrix products of the $n \times n$ matrix A with itself. By convention, $A^0 = I$. Consequently, all of the terms make good sense. They are all $n \times n$ matrices, so e^A is also an $n \times n$ matrix, provided that the series converges.

Convergence of this infinite series with matrix terms means that we consider the partial sum matrices

$$S_N = \sum_{k=0}^{N} \frac{1}{k!}A^k.$$

The components of S_N are very complicated expressions involving the entries of A. Convergence of the infinite series means that each component of the partial sum matrices converges. We will not worry about convergence. It is a fact, although we will not prove it, that the series in (1.6) converges for every matrix A. Furthermore, the convergence is rapid enough that all operations done on this series in what follows are justified.

EXAMPLE 1.7 ◆ Show that the exponential of the diagonal matrix $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ is the diagonal matrix $e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}$.

Since A is a diagonal matrix, the powers of A are easy to compute:

$$A^2 = A \cdot A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \cdot \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{pmatrix},$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{pmatrix} \cdot \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} r_1^3 & 0 \\ 0 & r_2^3 \end{pmatrix},$$

and so forth. Therefore,

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} r_1^3 & 0 \\ 0 & r_2^3 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} 1 + r_1 + r_1^2/2! + r_1^3/3! + \cdots & 0 \\ 0 & 1 + r_2 + r_2^2/2! + r_2^3/3! + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}. \end{aligned}$$

◆

Obviously, there is nothing special about the dimension 2 in this example. In general, the exponential of a diagonal matrix is the diagonal matrix containing the exponentials of the diagonal entries. In particular, we have

$$e^{rI} = e^r I. \tag{1.8}$$

When $r = 0$, we have the important special case

$$e^{0I} = e^0 I = I. \tag{1.9}$$

Soltuion to the intial value problem

We will be looking at

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots . \tag{1.10}$$

for a fixed $n \times n$ matrix A and a real number t. Consider e^{tA} as a function of t with values that are $n \times n$ matrices. If $\mathbf{v} \in \mathbf{R}^n$, we will also be looking at the function

$$e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \frac{t^3}{3!}A^3\mathbf{v} + \cdots , \tag{1.11}$$

which has values in $\mathbf{R}^n$. Remember that it is our goal to compute $e^{tA}\mathbf{v}$ for every $\mathbf{v}$ in a basis of $\mathbf{R}^n$.

Let's prove immediately that the exponential can be used to solve the initial value problem.

PROPOSITION 1.12 Suppose A is an $n \times n$ matrix.

1. Then
$$\frac{d}{dt} e^{tA} = Ae^{tA}.$$
2. If $\mathbf{v} \in \mathbf{R}^n$, the function $\mathbf{x}(t) = e^{tA}\mathbf{v}$ is the solution to the initial-value problem
$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}.$$

Proof Let's prove part (2) first, since it easily follows from part (1). First, by part (1),

$$\frac{d}{dt}\mathbf{x}(t) = \frac{d}{dt}\left(e^{tA}\mathbf{v}\right) = \frac{d}{dt}\left(e^{tA}\right)\mathbf{v} = Ae^{tA}\mathbf{v} = A\mathbf{x}(t).$$

To finish, we evaluate (1.11) at $t = 0$ to show that $\mathbf{x}(0) = \mathbf{v}$.

To prove part (1), we differentiate the series (1.10) term by term,

$$\begin{aligned}\frac{d}{dt}e^{tA} &= \frac{d}{dt}\left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)\\ &= A + \frac{t}{1!}A^2 + \frac{t^2}{2!}A^3 + \cdots\\ &= A\left(I + tA + \frac{t^2}{2!}A^2 + \cdots\right)\\ &= Ae^{tA}.\end{aligned}$$

The second part of Proposition 1.12 must seem like a panacea for readers who have been laboriously computing solutions to initial-value problems. It would be one, indeed, if the exponential of a matrix were easy to compute. However, the exponential of a matrix is usually difficult to compute.

Truncation

There is one other situation where we can easily compute the exponential $e^{tA}\mathbf{v}$. If most of the terms in the series in (1.11) are equal to the zero matrix, the infinite series becomes a finite sum. For example, if $A^2\mathbf{v} = \mathbf{0}$ and if $p > 2$, then $A^p\mathbf{v} = A^{p-2}A^2\mathbf{v} = A^{p-2}\mathbf{0} = \mathbf{0}$. Therefore, using (1.11) we have

$$\begin{aligned}e^{tA}\mathbf{v} &= \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \frac{t^3}{3!}A^3\mathbf{v} + \cdots\\ &= \mathbf{v} + tA\mathbf{v}.\end{aligned}$$

When this happens we will say that the series for $e^{tA}\mathbf{v}$ *truncates*. Since we will refer to it often, we will state the result as a proposition.

PROPOSITION 1.13 Suppose A is an $n \times n$ matrix and $\mathbf{v}$ is an n-vector.

1. If $A\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = \mathbf{v}$ for all t.
2. If $A^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v}$ for all t.
3. More generally, If $A^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}A^{k-1}\mathbf{v} \quad \text{for all } t \quad .$$

According to Proposition 1.13, we can compute $e^{tA}\mathbf{v}$ whenever the vector $\mathbf{v}$ is in the nullspace of A or in the nullspace of a power of A. Let's use the result to compute some examples.

EXAMPLE 1.14 ◆ Compute $e^{tA}\mathbf{v}$ where $A = \begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{v} = (-2, 1)^T$.

We compute that

$$A\mathbf{v} = \begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore by part (1) of Proposition 1.13, $e^{tA}\mathbf{v} = \mathbf{v}$. ◆

EXAMPLE 1.15 ◆ Consider

$$A = \begin{pmatrix} -4 & -2 & 1 \\ 4 & 2 & -2 \\ 8 & 4 & 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Compute $e^{tA}\mathbf{v}$ and $e^{tA}\mathbf{w}$.

We compute that

$$A\mathbf{v} = \begin{pmatrix} -4 & -2 & 1 \\ 4 & 2 & -2 \\ 8 & 4 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence by part (1) of Proposition 1.13, $e^{tA}\mathbf{v} = \mathbf{v}$. On the other hand,

$$A\mathbf{w} = \begin{pmatrix} -4 & -2 & 1 \\ 4 & 2 & -2 \\ 8 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = -\mathbf{v}.$$

so $A^2\mathbf{w} = \mathbf{0}$. Therefore by part (2) of Proposition 1.13,

$$\begin{aligned} e^{tA}\mathbf{w} &= \mathbf{w} + tA\mathbf{w} \\ &= \mathbf{w} - t\mathbf{v} \\ &= \begin{pmatrix} t \\ -2t \\ 1 \end{pmatrix}. \end{aligned}$$

◆

Proposition 1.13, together with the examples illustrating its use, provides a very modest beginning to the computations of $e^{tA}\mathbf{v}$. However, this modest beginning will bear fruit when we learn some properties of the exponential.

The law of exponents

The key property of the ordinary exponential function is the laaw of exponents

$$e^{a+b} = e^a e^b.$$

It's time for us to explore the extent to which this property remains true for the exponential of a matrix.

PROPOSITION 1.16

1. If A and B are $n \times n$ matrices, then

$$e^{A+B} = e^A e^B. \tag{1.17}$$

   if and only if $AB = BA$.
2. If A is an $n \times n$ matrix then e^A is a nonsingular matrix whose inverse is e^{-A}.

   If $AB = BA$ we say that A and B ***commute.***

Proof Part (1) is the most interesting property. It says that the usual law of exponents does not apply to matrices unless the two matrices commute. To prove it, we first compute $e^A e^B$ and regroup the terms as follows.

$$\begin{aligned} e^A e^B &= \left(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\right) \cdot \left(I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots\right) \\ &= I + (A + B) + \frac{1}{2!}(A^2 + 2AB + B^2) \qquad (1.18) \\ &\quad + \frac{1}{3!}(A^3 + 3A^2B + 3AB^2 + B^3) + \cdots \end{aligned}$$

On the other hand, e^{A+B} involves powers of $A + B$. We have

$$(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2.$$

Since $AB = BA$ (by assumption), $AB + BA = 2AB$. Therefore,

$$(A + B)^2 = A^2 + 2AB + B^2.$$

This is analogous to the familiar rule of squaring the sum of two numbers. Likewise, since A and B commute, we have

$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3.$$

With this information, we compute

$$\begin{aligned} e^{A+B} &= I + (A + B) + \frac{1}{2!}(A + B)^2 + \frac{1}{3!}(A + B)^3 + \cdots \\ &= I + (A + B) + \frac{1}{2!}\left(A^2 + 2AB + B^2\right) \\ &\quad + \frac{1}{3!}\left(A^3 + 3A^2B + 3AB^2 + B^3\right) + \cdots . \end{aligned}$$

The expression on the right is the same as the right side of (1.18). Thus, $e^{A+B} = e^A e^B$.

Part (2) follows easily from part (1). Since A and $-A$ commute,

$$\begin{aligned} e^A e^{-A} &= e^{A-A} \quad \text{from (1.17)} \\ &= e^{0I} \\ &= I \quad \text{from (1.9).} \end{aligned}$$

Thus, e^{-A} is the inverse of e^A.

Part (1) of Proposition 1.16 seems to be a departure from the behavior of the exponential function. In order for the law of exponents to hold, we need to assume that the two matrices commute. On the other hand, two numbers always commute, so perhaps the best way to interpret this new phenomenon is to realize that we needed no assumption for the exponential of a sum of numbers simply because two numbers always commute.

We can use part(1) of Proposition 1.16 together with Proposition 1.13 to greatly extend our capability of computing $e^{tA}\mathbf{v}$. Notice that if λ is a number, then

$$tA = \lambda tI + t[A - \lambda I]. \tag{1.19}$$

In addition, since the identity matrix commutes with every other matrix, the two summands in (1.19) commute. It follows from part (1) of Proposition 1.16 that

$$e^{tA} = e^{\lambda tI + t[A-\lambda I]} = e^{\lambda tI}e^{t[A-\lambda I]}. \tag{1.20}$$

Since the matrix λtI in the first factor in (1.20) is a diagonal matrix, we can compute that $e^{\lambda tI} = e^{\lambda t}I$, as we did in equation (1.8). Thus (1.20) becomes

$$e^{tA} = e^{\lambda t}Ie^{t[A-\lambda I]} = e^{\lambda t}e^{t[A-\lambda I]}. \tag{1.21}$$

When we apply (1.21) this to a vector $\mathbf{v}$ we get

$$e^{tA}\mathbf{v} = e^{\lambda t}e^{t[A-\lambda I]}\mathbf{v}. \tag{1.22}$$

Therefore, we can compute $e^{tA}\mathbf{v}$ if we can compute $e^{t[A-\lambda I]}\mathbf{v}$ for some number λ. In particular, according to Proposition 1.13, we can compute $e^{tA}\mathbf{v}$ if $\mathbf{v}$ is in the nullspace of $A - \lambda I$ for some number λ. Let's illustrate how equation (1.22) can be used in conjunction with Proposition 1.13.

EXAMPLE 1.23 ◆ Consider

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We compute that

$$A\mathbf{v}_1 = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\mathbf{v}_1.$$

Hence $[A - 2I]\mathbf{v}_1 = A\mathbf{v}_1 - 2\mathbf{v}_1 = \mathbf{0}$. By Proposition 1.13, $e^{t[A-2I]}\mathbf{v}_1 = \mathbf{v}_1$. Then by equation (1.22) with $\lambda = 2$,

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{2t}e^{A-2I}\mathbf{v}_1 = e^{2t}\mathbf{v}_1. \tag{1.24}$$

We also compute that $A\mathbf{v}_2 = 3\mathbf{v}_2$, or $[A - 3I]\mathbf{v}_1 = \mathbf{0}$. In exactly the same way, with $\lambda = 3$, we get

$$\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{-3t}e^{A-3I}\mathbf{v}_1 = e^{-3t}\mathbf{v}_1. \tag{1.25}$$

Notice that by Proposition 1.12, $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions to the initial value problems

$$\mathbf{x}_1' = A\mathbf{x}_1 \quad \text{with} \quad \mathbf{x}_1(0) = \mathbf{v}_1, \quad \text{and} \quad \mathbf{x}_2' = A\mathbf{x}_2 \quad \text{with} \quad \mathbf{x}_2(0) = \mathbf{v}_2.$$

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, the functions $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$. ◆

Example 1.23 is exciting, since we were able to find a fundamental set of solutions. The question is, how did we find numbers $\lambda = 2$ and 3 and the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in Example 1.23? Learning how to do that is our next problem.

Before doing that let's record the key idea in Example 1.23. Notice that we can equation (1.22) and Proposition 1.13 to prove:

PROPOSITION 1.26 Suppose A is an $n \times n$ matrix, λ is a number, and $\mathbf{v}$ is an n-vector.

1. If $[A - \lambda I]\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$ for all t.
2. If $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\,(\mathbf{v} + t[A - \lambda I]\mathbf{v})$ for all t.
3. More generally, if k is a positive integer and $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = e^{\lambda t}\left(\mathbf{v} + t[A - \lambda I]\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v}\right) \quad \text{for all } t.$$

9.2 Eigenvalues and eigenvectors

In this section we will exploit part (1) of Proposition 1.26. All by itself it is powerful enough to solve many systems of differential equations. The other parts of Proposition 1.26 will be needed to handle exceptional cases.

Let's look at the key idea in part (1) of Propostion 1.26. We were given a number λ and a vector $\mathbf{v}$ which satisfy $A\mathbf{v} = \lambda\mathbf{v}$. Then $[A - \lambda I]\mathbf{v} = \mathbf{0}$, so by Proposition 1.13, $e^{t[A-\lambda I]}\mathbf{v} = \mathbf{v}$. Finally, by equation (1.22) we have

$$e^{tA}\mathbf{v} = e^{\lambda t}e^{t[A-\lambda I]}\mathbf{v} = e^{\lambda t}\mathbf{v}.$$

Clearly, it is important to find numbers λ and vectors $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. The number λ and the associated vector $\mathbf{v}$ that satisfy this equation have special names.

DEFINITION 2.1 Suppose A is an $n \times n$ matrix. A number λ is called an ***eigenvalue*** of A if there is a nonzero vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{2.2}$$

If λ is an eigenvalue, then any vector $\mathbf{v}$ satisfying (2.2) is called an ***eigenvector*** associated with the eigenvalue λ.

The requirement in the definition of eigenvalue that $\mathbf{v}$ be nonzero is necessary, since the equation $A\mathbf{v} = \lambda\mathbf{v}$ always holds for $\mathbf{v} = \mathbf{0}$.

We have discovered that an eigenvalue-eigenvector pair always leads to a solution. Let's state this formally.

THEOREM 2.3 Suppose that λ is an eigenvalue of the matrix A and $\mathbf{v}$ is an associated eigenvector. Then

$$\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$$

is a solution to the system $\mathbf{x}' = A\mathbf{x}$ and satisfies the initial condition $\mathbf{x}(0) = \mathbf{v}$.

Solutions of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ found in Theorem 2.3 are called ***exponential solutions.***

Finding eigenvalues

Since finding eigenvalues and eigenvectors is so important, let's discuss techniques for computing them. We have nticed several times that

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Leftrightarrow \quad [A - \lambda I]\mathbf{v} = \mathbf{0}. \tag{2.4}$$

These equivalent formulas have different characters. We will emphasize the second. If $\mathbf{v}$ is a nonzero vector and $[A - \lambda I]\mathbf{v} = \mathbf{0}$, the matrix $A - \lambda I$ has a nontrivial nullspace. By Proposition 4.16 in Section 7.4, this can happen if and only if the matrix $A - \lambda I$ is singular. Let's summarize our thoughts.

PROPOSITION 2.5 Let A be an $n \times n$ matrix.

1. λ is an eigenvalue of A if and only if $A - \lambda I$ is singular.
2. If λ is an eigenvalue of A, then the set of all eigenvectors associated with λ is equal to the nullspace of $A - \lambda I$.

According to part (2) of Proposition 2.5 the set of eigenvectors associated to an eigenvalue is a subspace of $\mathbf{R}^n$. It is called the ***eigenspace*** of λ. Because it is a subspace, any linear combination of eigenvectors is again an eigenvector. In addition, a convenient way to specify an eigenspace is to find a basis for it.

By Corollary 6.3 in Section 7.6, $A - \lambda I$ is singular if and only if

$$\det(A - \lambda I) = 0. \tag{2.6}$$

The variable λ appears only in the diagonal entries of the matrix $A - \lambda I$. The determinant of $A - \lambda I$ can be written as a sum of terms, each of which is a product of the entries of $A - \lambda I$, with one from each row and one from each column. Hence, when we take the determinant in (2.6), we get a polynomial in λ, and if A is an $n \times n$ matrix, the highest power that can occur is n. Therefore,

$$p(\lambda) = \det(A - \lambda I)$$

is a polynomial of degree n. The polynomial $p(\lambda)$ is called the ***characteristic polynomial*** of A, and the equation

$$p(\lambda) = \det(A - \lambda I) = 0$$

is called the ***characteristic equation***. We have discovered an important fact. Let's state it formally.

PROPOSITION 2.7 The eigenvalues of an $n \times n$ matrix A are the roots of its characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

EXAMPLE 2.8 ◆ Find the eigenvalues of the matrix

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{pmatrix} \\ &= (-4-\lambda)(5-\lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1). \end{aligned}$$

Thus, the eigenvalues of A are 2 and -1. ◆

Since we know that the eigenvalues of A are the roots of the characteristic polynomial, we have made a significant step forward. To find the eigenvalues, we only have to find the roots of the characteristic polynomial. That is not always easy, but at least it is straightforward, and in difficult cases a computer can help. Notice also that to find the eigenvalues in this way, we do not at the same time have to find associated eigenvectors, as Definition 2.1 would seem to require.

Finding the eigenvectors

To find the eigenvectors associated to the eigenvalue λ, we use part (2) of Proposition 2.5 and look for the nullspace of $A - \lambda I$.

EXAMPLE 2.9 ◆ Find the eigenvectors for the matrix in Example 2.8.

For the eigenvalue 2, the eigenspace is the nullspace of

$$A - 2I = \begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}.$$

It is easily seen that the eigenspace is generated by the single vector $\mathbf{v}_1 = (1, 1)^T$. For the eigenvalue -1, the eigenspace is the nullspace of

$$A + I = \begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}.$$

It is easily seen that the eigenspace is generated by the vector $\mathbf{v}_2 = (2, 1)^T$. ◆

Let's take this example back to our original problem, solving systems of differential equations.

EXAMPLE 2.10 ◆ Find a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$ for the matrix A in Example 2.8.

We have done almost all of the work in Examples 2.8 and 2.9, where we computed the eigenvalues of A and the associated eigenvectors. We need only apply Theorem 2.3 to find the solutions. Using the previous two examples, we see that we have solutions

$$\mathbf{y}_1(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

These two solutions are linearly independent, since they are linearly independent for $t = 0$ (see Proposition 4.18 in Chapter 8). Consequently, they form a fundamental set of solutions. ◆

We can now explain how we got the numbers and vectors in Example 1.23.

EXAMPLE 2.11 ◆ Find a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$.

Notice that A is the matrix from Example 1.23. The characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\begin{pmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{pmatrix} \\ &= (1-\lambda)(4-\lambda) + 2 \\ &= \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 2)(\lambda - 3). \end{aligned}$$

The eigenvalues are the roots, $\lambda_1 = 2$ and $\lambda_2 = 3$. The eigenspace for $\lambda_1 = 2$ is the nullspace of

$$A - 2I = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix},$$

which is spanned by the single vector $\mathbf{v}_1 = (2, 1)^T$. The eigenspace for $\lambda_2 = 2$ is the nullspace of

$$A - 3I = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix},$$

which is spanned by the single vector $\mathbf{v}_2 = (1, 1)^T$. Therefore we get solutions

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $\mathbf{x}_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent, $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions. ◆

Summary

Let us examine how far this takes us to the completion of our strategy of finding n linearly independent solutions for a linear system of dimension n. The characteristic polynomial is of degree n. In general, a polynomial of degree n has n roots. Each root λ is an eigenvalue, and for each we can find a nonzero eigenvector $\mathbf{v}$. From these, we can form the exponential solution $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$. That's n solutions. If these are linearly independent (and we shall see that this is always the case if the eigenvalues are all different), we should be through. However, there are some complications that we will look into in the following sections.

1. **Distinct real roots:** We are essentially done here. The situation we examined in Examples 2.10 and 2.11 is what happens in general.
2. **Complex roots:** If an eigenvalue is complex, then the exponential solution is complex valued. Since we will usually want real-valued solutions, this is a complication that we will have to deal with.
3. **Repeated roots:** Sometimes the roots of a polynomial are not distinct. If that polynomial is a characteristic polynomial, then the number of distinct eigenvalues is strictly less than n. For each eigenvalue, we are guaranteed only one solution. Hence, our method will give us fewer solutions than we are looking for. This complication will also be dealt with in later sections, however, notice that we have not yet fully exploited Proposition 1.13.

9.3 Planar Systems

Before going to higher dimensional systems, let's look carefully at linear systems of dimension 2. These are also called ***planar systems***. In this section, we will do the algebra that will enable us to the solve the system

$$\mathbf{y}' = A\mathbf{y},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

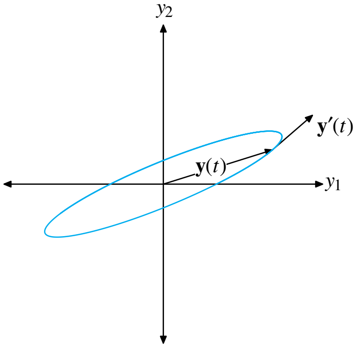


Figure 1 A solution curve in the phase plane.

We will find that some of this algebra applies equally well to higher dimensional systems, and when this is so, we will state the general result. This will prepare us for our examination of higher dimensional systems in Section 9.4.

Notice that as t varies, the solution curve $t \to \mathbf{y}(t)$ is a parametrically defined curve in the phase plane $\mathbf{R}^2$. An example is shown in Figure 1. We have discussed this in some detail in Section 8.2 of Chapter 8. In the next section, we look at the solution curves in the phase plane and carefully categorize the possible behaviors of solutions to planar systems.

According to the method discovered in the previous section, we want to look for exponential solutions of the form $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$, where λ is an eigenvalue of A and $\mathbf{v}$ is an associated eigenvector. The eigenvalues are solutions of the characteristic equation $\det(A - \lambda I) = 0$. Let's expand this in terms of the entries of A,

$$\begin{aligned}\det(A - \lambda I) &= \det\begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}).\end{aligned}$$

This is a quadratic polynomial. The constant term will be recognized as the determinant of A. We will denote this by

$$D = \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

The coefficient of λ is $T = a_{11} + a_{22}$. Notice that T is the sum of the diagonal entries of the matrix A. For a matrix A of any dimension, the ***trace*** is defined to be the sum of the diagonal elements, and it is denoted by $\mathrm{tr}(A)$. Thus, we have

$$T = \mathrm{tr}(A) = a_{11} + a_{22}.$$

With these definitions, the characteristic equation of the planar system becomes

$$\lambda^2 - T\lambda + D = 0. \tag{3.1}$$

The eigenvalues of A are the roots of the characteristic polynomial and are given by

$$\lambda = \left(T \pm \sqrt{T^2 - 4D}\right)/2. \tag{3.2}$$

There are three cases we must consider:

1. two distinct real roots (when $T^2 - 4D > 0$)
2. two complex conjugate roots (when $T^2 - 4D < 0$)
3. one real root of multiplicity 2 (when $T^2 - 4D = 0$)

Before doing so, let's prove a result that will make proving the independence of solutions very easy.

PROPOSITION 3.3 Suppose λ_1 and λ_2 are eigenvalues of an $n \times n$ matrix A. Suppose $\mathbf{v}_1 \neq \mathbf{0}$ is an eigenvector for λ_1 and $\mathbf{v}_2 \neq \mathbf{0}$ is an eigenvector for λ_2. If $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Proof Suppose there are constants c_1 and c_2 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}. \tag{3.4}$$

To show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, we need to show that c_1 and c_2 are both equal to 0.

Multiply (3.4) by the matrix A. Since $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, we get

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 = \mathbf{0}. \tag{3.5}$$

Multiply (3.4) by λ_2 and subtract from (3.5) to get

$$c_1(\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0}. \tag{3.6}$$

It is our assumption that $\lambda_1 \neq \lambda_2$, so $\lambda_1 - \lambda_2 \neq 0$. In addition, the eigenvector $\mathbf{v}_1 \neq \mathbf{0}$. Hence, we must have $c_1 = 0$. We can prove that $c_2 = 0$ in a similar manner.

Note that it is not necessary that the eigenvalues and the eigenvectors be real in Proposition 3.3. If one or more is complex, the proposition is still true.

Distinct real eigenvalues

This is the case when $T^2 - 4D > 0$. The solutions of the characteristic equation (3.1) are

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} \quad \text{and} \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}.$$

Then $\lambda_1 < \lambda_2$, and both are real eigenvalues of A. Notice that Examples 2.10 and 2.11 in Section 9.1 are of this type.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be associated eigenvectors. Then we have two exponential solutions

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \quad \mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_2 t}\mathbf{v}_2.$$

According to Proposition 3.3, $\mathbf{y}_1(0) = \mathbf{v}_1$ and $\mathbf{y}_2(0) = \mathbf{v}_2$ are linearly independent. Consequently, by Proposition 4.18 in Section 8.4, the solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent and form a fundamental set of solutions. The general solution is

$$\mathbf{y}(t) = C_1\mathbf{y}_1(t) + C_2\mathbf{y}_2(t).$$

Let's summarize this in a theorem.

THEOREM 3.7 Suppose that A is a 2×2 matrix with real eigenvalues $\lambda_1 \neq \lambda_2$. Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors associated with the eigenvalues.

1. A fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$ is given by $\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_2 t}\mathbf{v}_2$.
2. The general solution is

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2,$$

where C_1 and C_2 are arbitrary constants.

EXAMPLE 3.8 ◆ Find the general solution to the system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}.$$

A has characteristic polynomial $\lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1)$. Thus, the eigenvalues are -1 and -2. The eigenspace for the eigenvalue -1 is the nullspace of the matrix

$$A - (-1)I = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix},$$

which is spanned by the vector $(2, 3)^T$. Hence, we have the solution

$$\mathbf{y}_1(t) = e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

The eigenspace for -2 is the nullspace of

$$A-(-2)I=\begin{pmatrix}-2 & 2\\ -3 & 3\end{pmatrix}.$$

This time $(1,1)^T$ is a basis, so

$$\mathbf{y}_2(t)=e^{-2t}\begin{pmatrix}1\\ 1\end{pmatrix}$$

is a solution. According to Theorem 3.7, these solutions are linearly independent. Thus, $\mathbf{y}_1$ and $\mathbf{y}_2$ are a fundamental set of solutions, and the general solution is

$$\mathbf{y}(t)=C_1e^{-t}\begin{pmatrix}2\\ 3\end{pmatrix}+C_2e^{-2t}\begin{pmatrix}1\\ 1\end{pmatrix}.$$

◆

Complex matrices

When we look for the complex eigenvalues of a matrix and the associated eigenvectors, we are naturally led to consider matrices and vectors with complex entries, or complex matrices and vectors. It is appropriate to say a few words about such objects. As a general statement, most operations on complex numbers transfer with little or no change to operations on complex vectors and matrices. We discussed complex numbers and their properties in Section 4.3 in Chapter 4, and you should reread that section if you need to refresh your knowledge of the subject.

An example of a complex matrix is

$$M=\begin{pmatrix}4+5i & 3-2i & -3i\\ 2+i & 4 & 0\end{pmatrix}. \tag{3.9}$$

Any complex matrix can be split into its real and imaginary parts in exactly the same way as complex numbers are split. We have

$$M=A+iB,\quad \text{where}\quad A=\mathrm{Re}(M)\quad\text{and}\quad B=\mathrm{Im}(M). \tag{3.10}$$

For the example in (3.9), we have

$$A=\mathrm{Re}(M)=\begin{pmatrix}4 & 3 & 0\\ 2 & 4 & 0\end{pmatrix}\quad\text{and}\quad B=\mathrm{Im}(M)=\begin{pmatrix}5 & -2 & -3\\ 1 & 0 & 0\end{pmatrix}.$$

The complex conjugate of a complex matrix is computed component by component. Thus, for the matrix in (3.9), we have

$$\overline{M}=\begin{pmatrix}\overline{4+5i} & \overline{3-2i} & \overline{-3i}\\ \overline{2+i} & \overline{4} & \overline{0}\end{pmatrix}=\begin{pmatrix}4-5i & 3+2i & 3i\\ 2-i & 4 & 0\end{pmatrix}.$$

If $M=A+iB$, where $A=\mathrm{Re}(M)$ and $B=\mathrm{Im}(M)$, then $\overline{M}=A-iB$, just as though the matrices were complex numbers. From these considerations, we find that

$$M \text{ is a real matrix if and only if } \overline{M}=M, \tag{3.11}$$

and

$$\operatorname{Re}(M) = \frac{1}{2}(M + \overline{M}); \quad \operatorname{Im}(M) = \frac{1}{2i}(M - \overline{M}). \tag{3.12}$$

Finally, the operation of conjugation behaves well with the matrix operations of addition and multiplication. If M and N are complex matrices, then

$$\overline{M + N} = \overline{M} + \overline{N}. \tag{3.13}$$

If $\mathbf{z}$ is a complex vector, then

$$\overline{M\mathbf{z}} = \overline{M}\,\overline{\mathbf{z}}. \tag{3.14}$$

Finally if α is a complex number, then

$$\overline{\alpha\mathbf{z}} = \overline{\alpha}\,\overline{\mathbf{z}}. \tag{3.15}$$

Complex eigenvalues

This is the case when $T^2 - 4D < 0$. The roots of the characteristic equation are the complex conjugates

$$\lambda = \frac{T + i\sqrt{4D - T^2}}{2} \quad \text{and} \quad \overline{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}. \tag{3.16}$$

Let's look at an example.

EXAMPLE 3.17 ◆ Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}.$$

The eigenvalues are solutions to the characteristic equation

$$0 = \det(A - \lambda I) = \lambda^2 - 2\lambda + 2.$$

By the quadratic formula, the roots of the equation $\lambda^2 - 2\lambda + 2 = 0$ are the complex conjugates $\lambda = 1 + i$ and $\overline{\lambda} = 1 - i$.

To find an eigenvector for $\lambda = 1 + i$, we look for vectors in the nullspace of the complex matrix

$$A - \lambda I = A - (1+i)I = \begin{pmatrix} -(1+i) & 1 \\ -2 & 2 - (1+i) \end{pmatrix} = \begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix}.$$

The standard methods of finding the nullspace of a matrix work, even though some of the coefficients are complex. Do not be put off because this matrix does not *look* singular. Remember, we chose the eigenvalue λ to make this matrix singular, so it had better be singular. Let's have a little faith in our work up to now and try to find a vector in the nullspace by finding a vector that is killed by the first row of the matrix. We want a vector $\mathbf{w} = (w_1, w_2)^T$ such that $-(1+i)w_1 + w_2 = 0$. Clearly,

$$\mathbf{w} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \tag{3.18}$$

is such a vector. To reassure yourself, you might show directly that

$$A\mathbf{w} = \lambda\mathbf{w}. \tag{3.19}$$

Thus, although it has complex entries, $\mathbf{w}$ is an eigenvector associated to $\lambda = 1 + i$.

We could repeat this process to find the eigenspace for the second eigenvalue $\overline{\lambda} = 1 - i$, but there is a better way that works in general. If we conjugate equation (3.19), we get

$$\overline{A\mathbf{w}} = \overline{\lambda\mathbf{w}}. \tag{3.20}$$

Since A is a real matrix, the left-hand side is $\overline{A\mathbf{w}} = \overline{A}\,\overline{\mathbf{w}} = A\overline{\mathbf{w}}$, by (3.14) and (3.11). The right-hand side is $\overline{\lambda\mathbf{w}} = \overline{\lambda}\,\overline{\mathbf{w}}$ by (3.15). Thus, (3.20) becomes

$$A\overline{\mathbf{w}} = \overline{\lambda}\,\overline{\mathbf{w}}. \tag{3.21}$$

Thus, the complex conjugate of $\mathbf{w}$,

$$\overline{\mathbf{w}} = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix},$$

is an eigenvector associated to $\overline{\lambda} = 1 - i$. ◆

Let's continue with the matrix in Example 3.17. We get solutions to $\mathbf{x}' = A\mathbf{x}$ in the way indicated in Theorem 2.3. Corresponding to the eigenvalue $\lambda = 1 + i$, we have the solution

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} = e^{(1+i)t}\begin{pmatrix} 1 \\ 1 + i \end{pmatrix}. \tag{3.22}$$

Since $\overline{\mathbf{w}}$ is an eigenvector associated with the eigenvalue $\overline{\lambda}$, we get the complex exponential solution $e^{\overline{\lambda}t}\,\overline{\mathbf{w}}$. However, using Proposition 3.15 of Section 4.3, we see that

$$\overline{\mathbf{z}}(t) = \overline{e^{\lambda t}\mathbf{w}} = e^{\overline{\lambda}t}\,\overline{\mathbf{w}}.$$

Thus, the solution corresponding to $\overline{\lambda}$ is the complex conjugate of $\mathbf{z}$,

$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda}t}\,\overline{\mathbf{w}} = e^{(1-i)t}\begin{pmatrix} 1 \\ 1 - i \end{pmatrix}. \tag{3.23}$$

The method we used to find the solutions in (3.22) and (3.23) is completely general. Suppose we have a matrix A with complex conjugate eigenvalues λ and $\overline{\lambda}$, as in (3.16). Suppose that $\mathbf{w}$ is an eigenvector associated with λ. Then its complex conjugate $\overline{\mathbf{w}}$ is an eigenvector corresponding to $\overline{\lambda}$. The argument proving this in general was given in going from (3.19) to (3.21).

Notice also that $\mathbf{w}$ and $\overline{\mathbf{w}}$ are eigenvectors associated with different eigenvalues. By Proposition 3.3, they are linearly independent, and therefore $\mathbf{z}$ and $\overline{\mathbf{z}}$ form a fundamental system of solutions. This argument applies in the general situation, so we have proved the following theorem.

THEOREM 3.24 Suppose that A is a 2×2 matrix with complex conjugate eigenvalues λ and $\overline{\lambda}$. Suppose that $\mathbf{w}$ is an eigenvector associated with λ.

1. A fundamental set of complex valued solutions to the system $\mathbf{y}' = A\mathbf{y}$ is given by

$$\mathbf{z}(t) = e^{tA}\mathbf{w} = e^{\lambda t}\mathbf{w} \quad \text{and} \quad \overline{\mathbf{z}}(t) = e^{tA}\overline{\mathbf{w}} = e^{\overline{\lambda} t}\overline{\mathbf{w}}.$$

2. The general solution to the system $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}(t) = C_1 e^{\lambda t}\mathbf{w} + C_2 e^{\overline{\lambda} t}\,\overline{\mathbf{w}},$$

where C_1 and C_2 are arbitrary constants.

The solutions in Theorem 3.24 are complex valued. Complex-valued solutions are preferred in some situations (for example, in electrical engineering and physics). However, in other situations, it is important to find real-valued solutions. Fortunately, the real and imaginary parts of a complex solution provide the needed fundamental set of solutions.

PROPOSITION 3.25 Suppose A is an $n \times n$ matrix with real coefficients, and suppose that $\mathbf{z}(t) = \mathbf{x}_1(t) + i\mathbf{x}_2(t)$ is a solution to the system

$$\mathbf{z}' = A\mathbf{z}. \tag{3.26}$$

(a) The complex conjugate $\overline{\mathbf{z}} = \mathbf{x}_1 - i\mathbf{x}_2$ is also a solution to (3.26).

(b) The real and imaginary parts $\mathbf{x}_1$ and $\mathbf{x}_2$ are also solutions to (3.26). Furthermore, if $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent, so are $\mathbf{x}_1$ and $\mathbf{x}_2$.

Proof To prove (a), we just conjugate (3.26) and remember that since A has real entries, $\overline{A} = A$. Hence

$$\overline{\mathbf{z}}' = \overline{\mathbf{z}'} = \overline{A\mathbf{z}} = \overline{A}\,\overline{\mathbf{z}} = A\overline{\mathbf{z}}.$$

The proof of part (b) is revealing. If we look at the sum of $\mathbf{z}$ and $\overline{\mathbf{z}}$ and then at their difference, we get

$$\mathbf{x}_1 = \frac{1}{2}(\mathbf{z} + \overline{\mathbf{z}}) \quad \text{and} \quad \mathbf{x}_2 = \frac{1}{2i}(\mathbf{z} - \overline{\mathbf{z}}). \tag{3.27}$$

Thus, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linear combinations of $\mathbf{z}$ and $\overline{\mathbf{z}}$. According to Theorem 4.8 in Chapter 8, they are solutions to (3.26).

To show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, suppose the contrary, that they are dependent. Then there is a constant c such that $\mathbf{x}_2 = c\mathbf{x}_1$, so $\mathbf{z} = (1 + ic)\mathbf{x}_1$ and $\overline{\mathbf{z}} = (1 - ic)\mathbf{x}_1$. This means that $(1 - ic)\mathbf{z} - (1 + ic)\overline{\mathbf{z}} = 0$, which implies that $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly dependent, contradicting our assumption.

If A is a real matrix, the exponential matrix e^{tA} is also a real matrix. The initial value problem $\mathbf{z}' = A\mathbf{z}$ with $\mathbf{z}(0) = \mathbf{w}$, where $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ has solution

$$\mathbf{z}(t) = e^{tA}\mathbf{w} = e^{tA}[\mathbf{v}_1 + i\mathbf{v}_2] = e^{tA}\mathbf{v}_1 + ie^{tA}\mathbf{v}_2.$$

On the other hand, $\mathbf{z}$ has real and imaginary parts $\mathbf{z} = \mathbf{x}_1 + i\mathbf{x}_2$. Equating real and imaginary parts, we find that

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 \quad \text{and} \quad \mathbf{x}_2(t) = e^{tA}\mathbf{v}_2.$$

If A has a complex eigenvalue $\lambda = \alpha + i\beta$ and associated eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$, we have $\mathbf{z}(t) = e^{tA}\mathbf{w} = e^{\lambda t}\mathbf{w}$. A little algebra using Euler's formula finds the real and imaginary parts.

$$\begin{aligned}
\mathbf{z}(t) &= e^{\lambda t}\mathbf{w} \\
&= e^{(\alpha+i\beta)t}(\mathbf{v}_1 + i\mathbf{v}_2) \\
&= e^{\alpha t}\,(\cos\beta t + i\sin\beta t)\,(\mathbf{v}_1 + i\mathbf{v}_2) \\
&= e^{\alpha t}\,(\cos\beta t\,\mathbf{v}_1 - \sin\beta t\,\mathbf{v}_2) + ie^{\alpha t}\,(\sin\beta t\,\mathbf{v}_1 + \cos\beta t\,\mathbf{v}_2)
\end{aligned}$$

Thus the real and imaginary parts of the solution $\mathbf{z}$ are

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{tA}\mathbf{v}_1 = e^{\alpha t}\,(\cos\beta t\,\mathbf{v}_1 - \sin\beta t\,\mathbf{v}_2) \quad \text{and} \\
\mathbf{x}_2(t) &= e^{tA}\mathbf{v}_2 = e^{\alpha t}\,(\sin\beta t\,\mathbf{v}_1 + \cos\beta t\,\mathbf{v}_2)
\end{aligned}$$

This leads immediately to the following theorem.

THEOREM 3.28 Suppose that A is a 2×2 matrix with a complex eigenvalue $\lambda = \alpha + i\beta$ and associated eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$.

1. A fundamental set of real solutions to $\mathbf{x}' = A\mathbf{x}$ is given by

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{tA}\mathbf{v}_1 = e^{\alpha t}\,(\cos\beta t\,\mathbf{v}_1 - \sin\beta t\,\mathbf{v}_2) \quad \text{and} \\
\mathbf{x}_2(t) &= e^{tA}\mathbf{v}_2 = e^{\alpha t}\,(\sin\beta t\,\mathbf{v}_1 + \cos\beta t\,\mathbf{v}_2)
\end{aligned}$$

2. The general solution of the system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = C_1e^{\alpha t}\,(\cos\beta t\,\mathbf{v}_1 - \sin\beta t\,\mathbf{v}_2) + C_2e^{\alpha t}\,(\sin\beta t\,\mathbf{v}_1 + \cos\beta t\,\mathbf{v}_2)\,,$$

where C_1 and C_2 are arbitrary constants.

Let's return to the system in Example 3.17.

EXAMPLE 3.29 ◆ Find a fundamental set of real solutions for the system $\mathbf{y}' = A\mathbf{y}$, where A is the matrix defined in Example 3.17.

We have already determined the two complex-valued solutions in (3.22) and (3.23). Since $\mathbf{z}(0) = \mathbf{w}$ and $\overline{\mathbf{z}}(0) = \overline{\mathbf{w}}$, which are eigenvectors corresponding to the eigenvalues λ and $\overline{\lambda}$, respectively, we know that $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent. Motivated by Proposition 3.25, let's break $\mathbf{z}(t)$ into its real and imaginary parts. Using Euler's formula,

$$\begin{aligned}
\mathbf{z}(t) &= e^{(1+i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} \\
&= e^{t}\,(\cos t + i\sin t)\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\
&= e^{t}\left\{\left(\cos t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + i\left(\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\right\} \\
&= e^{t}\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + ie^{t}\begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix}.
\end{aligned} \tag{3.30}$$

According to Proposition 3.25, the real and imaginary parts of $\mathbf{z}$ are also solutions. From (3.30), we see that these are

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} \quad \text{and}$$

$$\mathbf{x}_2(t) = e^t \begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix}.$$

Again according to Proposition 3.25, these solutions are linearly independent, since $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent. ◆

We could have found the answer in Example 3.29 by plugging into the formula in Theorem 3.28, but it is usually better to remember how to use Euler's formula and complex arithmetic to find the answers. The formula in Theorem 3.28 is useful primarily for understanding the form the general solution takes.

One real eigenvalue of multiplicity 2

This is the degenerate case when $T^2 = 4D$. The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - T\lambda + T^2/4 = (\lambda - T/2)^2 = (\lambda - \lambda_1)^2, \qquad (3.31)$$

where $\lambda_1 = T/2$ is the only eigenvalue and it has multiplicity 2. There are two subcases, depending on the dimension of the eigenspace of λ. Since the eigenspace is a subspace of $\mathbf{R}^2$, it can have dimension 1 or 2.

If the eigenspace has dimension 2, it must be equal to all of $\mathbf{R}^2$. In this case, every vector is an eigenvector, so $A\mathbf{v} = \lambda_1\mathbf{v}$ for all $\mathbf{v} \in \mathbf{R}^2$. This can only happen if $A = \lambda_1 I$. Hence, the system has the very simple form

$$x_1' = \lambda_1 x_1,$$
$$x_2' = \lambda_1 x_2,$$

or $\mathbf{x}' = \lambda_1\mathbf{x}$. For any vector $\mathbf{v}$, the solution with initial value $\mathbf{x}(0) = \mathbf{v}$ is $\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{v}$—very easy indeed.

The more interesting case is when the dimension of the eigenspace is 1. Let's illustrate the problem with an example.

EXAMPLE 3.32 ◆ Let

$$A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}.$$

Find all exponential solutions to the system $\mathbf{x}' = A\mathbf{x}$.

The characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2.$$

The only root, and therefore the only eigenvalue of A, is $\lambda_1 = -1$. We have

$$A - \lambda_1 I = A + I = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}.$$

The vector $\mathbf{v}_1 = (2, -1)^T$ is a basis for the eigenspace, the nullspace of $A + I$. The corresponding solution is

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Since the dimension of the eigenspace of $\lambda_1 = -1$ is 1, this exhausts the possibilities for exponential solutions. Any exponential solution must be a constant multiple of $\mathbf{x}_1$. ◆

Since we can find only one linearly independent exponential solution to the system in Example 3.32, we must try something else to get a second solution that is not a multiple of $\mathbf{x}_1(t)$. Remembering the parts of Proposition 1.26 that we have not yet used, we compute

$$[A - \lambda_1 I]^2 = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} = 0I. \tag{3.33}$$

Consequently, we can use part (2) of Proposition 1.26 to compute

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda_1 t}\left(\mathbf{v} + t[A - \lambda_1 I]\mathbf{v}\right) \\ &= e^{\lambda_1 t}\left(I + t[A - \lambda_1 t I]\right)\mathbf{v}, \end{aligned}$$

for all $\mathbf{v} \in \mathbf{R}^2$. It follows that

$$e^{tA} = e^{\lambda_1 t}\left(I + t[A - \lambda_1 t I]\right) = e^{-t}\begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}.$$

If we want a fundamental set of solutions we need only solve the initial value problem with two linearly independent initial values. For example, we could use $\mathbf{e}_1 = (1, 0)^T$ and $\mathbf{e}_2 = (0, 1)^T$. With this choice for the matrix in Example 3.32 we get the fundamental set

$$\begin{aligned} \mathbf{y}_1(t) &= e^{tA}\mathbf{e}_1 = e^{-t}\begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-t}\begin{pmatrix} 1 + 2t \\ -t \end{pmatrix} \quad \text{and} \\ \mathbf{y}_2(t) &= e^{tA}\mathbf{e}_2 = e^{-t}\begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{-t}\begin{pmatrix} 4t \\ 1 - 2t \end{pmatrix}. \end{aligned}$$

Notice that $\mathbf{y}_1$ is the first column of the matrix e^{tA} and $\mathbf{y}_2$ is the second column. Thus, once e^{tA} is computed, finding $\mathbf{y}_1$ and $\mathbf{y}_2$ is very easy.

We are successful in this case because A satisfies $(A - \lambda_1 I)^2 = 0I$, as we found in (3.33). This is not a fluke. It happens for any 2×2 matrix with one real eigenvalue of multiplicity 2. This follows from the Cayley-Hamilton theorem, which says that any matrix satisfies its characteristic equation. If A has one real eigenvalue λ_1 of multiplicity 2, then, by (3.31), its characteristic polynomial is $p(\lambda) = (\lambda - \lambda_1)^2$. To compute $p(A)$, we replace the number λ by the matrix A (including λ^2 by A^2 and $1 = \lambda^0$ with $A^0 = I$). The result is

$$p(A) = (A - \lambda_1 I)^2 = A^2 - 2\lambda_1 A + \lambda_1^2 A^0 = 0I. \tag{3.34}$$

For the matrix in Example 3.32, we verified this in (3.33).

We can compute the exponential of any matrix satisfying(3.34). We go back to equation (1.20), use the definition of the exponential, and use (3.34) to truncate the series and compute that

$$\begin{aligned} e^{tA} &= e^{\lambda_1 t} e^{t[A-\lambda_1 tI]} \\ &= e^{\lambda_1 t}\left(I + t[A-\lambda_1 tI] + \frac{t^2}{2!}[A-\lambda_1 tI]^2 + \cdots\right) \\ &= e^{\lambda_1 t}\left(I + t[A-\lambda_1 tI]\right) \end{aligned} \tag{3.35}$$

Having computed e^{tA}, we can solve any initial value problem using Theorem 2.3. In particular, we can use the initial conditions $\mathbf{e}_1 = (1, 0)^T$ and $\mathbf{e}_2 = (0, 1)^T$. Just as before, this choice leads to the solutions $\mathbf{y}_1$ and $\mathbf{y}_2$, where $\mathbf{y}_1$ is the first column of the matrix e^{tA} and $\mathbf{y}_2$ is the second column.

However, for some purposes we want to use a basis of initial conditions that includes an eigenvector of the matrix A. Suppose that A has the single eigenvalue λ_1 and that the eigenspace has dimension 1 with basis $\mathbf{v}_1$. For the second vector in the basis of $\mathbf{R}^2$ we could choose any vector that is not a multiple of $\mathbf{v}_1$, but we will choose $\mathbf{v}_2$ satisfying

$$[A-\lambda_1 I]\mathbf{v}_2 = \mathbf{v}_1. \tag{3.36}$$

We can solve this equation because A satisfies (3.34). To do so, we start with any vector $\mathbf{w}$ in $\mathbf{R}^2$ which is not a multiple of $\mathbf{v}_1$. Because A satisfies (3.34), we know that $[A-\lambda_1 I]^2\mathbf{w} = \mathbf{0}$. Hence $[A-\lambda_1 I]\mathbf{w}$ is in the nullspace of $A-\lambda_1 I$, which means that $[A-\lambda_1 I]\mathbf{w}$ is an eigenvector associated to the eigenvalue λ_1. Since the eigenspace for λ_1 has dimension 1, and $\mathbf{v}_1$ is a nonzero eigenvector, it follows that $[A-\lambda_1 I]\mathbf{w}$ is a multiple of $\mathbf{v}_1$. Hence, for any vector $\mathbf{w}$ in $\mathbf{R}^2$, there is a constant a such that

$$[A-\lambda_1 I]\mathbf{w} = a\mathbf{v}_1. \tag{3.37}$$

This means that with any vector $\mathbf{w}$, we are close to finding a solution to (3.36). To find a vector for which $a = 1$ in (3.37), we notice that if $\mathbf{w}$ is not a multiple of $\mathbf{v}_1$, then $\mathbf{w}$ is not an eigenvector. Therefore, $[A-\lambda_1 I]\mathbf{w} \neq \mathbf{0}$, so $a \neq 0$. We can set

$$\mathbf{v}_2 = (1/a)\mathbf{w}. \tag{3.38}$$

Then

$$[A-\lambda_1 I]\mathbf{v}_2 = \frac{1}{a}[A-\lambda_1 I]\mathbf{w} = \mathbf{v}_1,$$

and we have a solution to (3.36).

The solutions with initial values $\mathbf{v}_1$ and $\mathbf{v}_2$ are given by

$$\begin{aligned} &\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \\ &\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_1 t}\left(\mathbf{v}_2 + t[A-\lambda_1]\mathbf{v}_2\right) = e^{\lambda_1 t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right). \end{aligned}$$

Since we chose $\mathbf{v}_2$ so that it was not a multiple of $\mathbf{v}_1$, the solutions are linearly independent, and $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions.

Let's summarize the method for finding a fundamental set of solutions in a theorem.

THEOREM 3.39 Suppose that A is a 2×2 matrix with one eigenvalue λ_1 of multiplicity 2, and suppose that the eigenspace of λ has dimension 1.

1. $e^{tA} = e^{\lambda_1 t}\left(I + t[A - \lambda_1 I]\right)$.
2. The columns of e^{tA} form a fundamental set of solutions to the system $\mathbf{x}' = A\mathbf{x}$.
3. If $\mathbf{v}_1$ is a nonzero eigenvector, and $\mathbf{v}_2$ satisfies $[A - \lambda_1 I]\mathbf{v}_2 = \mathbf{v}_1$. Then

$$\begin{aligned} \mathbf{x}_1(t) &= e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \\ \mathbf{x}_2(t) &= e^{tA}\mathbf{v}_2 = e^{\lambda_1 t}[\mathbf{v}_2 + t\mathbf{v}_1] \end{aligned} \tag{3.40}$$

form a fundamental set of solutions to the system $\mathbf{x}' = A\mathbf{x}$.

Given the fundamental set of solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, we can write the general solution as

$$\begin{aligned} \mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \\ &= C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_1 t}[\mathbf{v}_2 + t\mathbf{v}_1] \\ &= e^{\lambda_1 t}[(C_1 + C_2 t)\mathbf{v}_1 + C_2\mathbf{v}_2]. \end{aligned} \tag{3.41}$$

EXAMPLE 3.42 ◆ Let

$$A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}.$$

Find a fundamental set of solutions to $\mathbf{x}' = A\mathbf{x}$.

This is the same system we looked at in Example 3.32. There we found that A has a single eigenvalue $\lambda_1 = -1$ and the eigenvector $\mathbf{v}_1 = (2, -1)^T$. The corresponding solution is

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

We follow Theorem 3.39 to find another solution. We need to find a vector $\mathbf{v}_2$ which satisfies $[A - \lambda_1 I]\mathbf{v}_2 = \mathbf{v}_1$. The process for doing this is covered in the paragraph containing formula (3.38). We can start with literally any vector $\mathbf{w}$ that is not a multiple of $\mathbf{v}_1$. Given this flexibility, we should choose a vector that contains as many zeros as possible to facilitate computation. For example, if we choose a simple vector like $\mathbf{w} = (0, 1)^T$, we proceed as follows. Compute

$$[A - \lambda_1 I]\mathbf{w} = [A + I]\mathbf{w} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = 2\mathbf{v}_1.$$

Hence, we take $\mathbf{v}_2 = (1/2)\mathbf{w} = (0, 1/2)^T$.

Our fundamental set of solutions is

$$\begin{aligned} \mathbf{x}_1(t) &= e^{tA}\mathbf{v}_1 = e^{\lambda t}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \\ \mathbf{x}_2(t) &= e^{tA}\mathbf{v}_2 = e^{\lambda t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right) = e^{-t}\begin{pmatrix} 2t \\ 1/2 - t \end{pmatrix}. \end{aligned}$$

The general solution is

$$\begin{aligned}\mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)\\ &= e^{-t}\left(C_1\begin{pmatrix}2\\-1\end{pmatrix} + C_2\begin{pmatrix}2t\\1/2-t\end{pmatrix}\right)\\ &= e^{-t}\left((C_1+C_2t)\begin{pmatrix}2\\-1\end{pmatrix} + C_2\begin{pmatrix}0\\1/2\end{pmatrix}\right).\end{aligned} \tag{3.43}$$

◆

9.4 Phase Plane Portraits

Now that we know how to solve planar systems, let's find out what the solutions look like. There is a variety of different cases, with different behaviors. We will examine the six most important cases here. There are several more. Some of them will be the subject of exercises, and a complete classification will be left as a project.

To set the stage, we will be considering the system

$$\mathbf{y}' = A\mathbf{y}, \tag{4.1}$$

where

$$A = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \quad \text{and} \quad \mathbf{y}(t) = \begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}.$$

The characteristic polynomial is

$$\lambda^2 - T\lambda + D = 0,$$

where $D = \det(A)$, and $T = \operatorname{tr}(A)$.

Since the system in (4.1) is autonomous, it is natural to use the phase plane to visualize the solutions. Review Section 8.3 of Chapter 8, if necessary, where we pointed out that the uniqueness theorem implies that solution curves cannot intersect. This is an important point to remember as we proceed.

We will find that the solutions have strong connections to the equilibrium points of the system. For the linear homogeneous system in (4.1), the equilibrium points are those points $\mathbf{v} \in \mathbf{R}^2$ where $A\mathbf{v} = \mathbf{0}$. Thus, the set of equilibrium points is just the nullspace of A. In most of the cases that we will examine, A is nonsingular, so the origin $\mathbf{0}$ is the only equilibrium point. There will be times when A is singular, in which case the nullspace is a line in $\mathbf{R}^2$, or all of $\mathbf{R}^2$, and every point in the nullspace is an equilibrium point.

Each of the cases will have a descriptive name, like *saddle point* or *center*. This name is meant to refer to the equilibrium point at the origin. Thus, for example, we will say that the origin is a saddle point.

Real eigenvalues

In the first three cases that we will consider, the eigenvalues of A will be real and distinct. For this to be true, we must have $T^2 - D > 0$. The eigenvalues are

$$\lambda_1 = \frac{T - \sqrt{T^2-4D}}{2} \quad \text{and} \quad \lambda_2 = \frac{T + \sqrt{T^2-4D}}{2}.$$

Notice that $\lambda_1 < \lambda_2$.

According to Theorem 3.7, the general solution is

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2, \tag{4.2}$$

where C_1 and C_2 are arbitrary constants and $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors associated with λ_1 and λ_2, respectively. Particular solutions are captured by assigning various values to the constants C_1 and C_2. Two particular solutions are especially noteworthy: the so-called ***exponential solutions***. These are the solutions $C_1 e^{\lambda_1 t}\mathbf{v}_1$ and $C_2 e^{\lambda_2 t}\mathbf{v}_2$.

Saddle point

Suppose that the eigenvalues are real and have different signs, so $\lambda_1 < 0 < \lambda_2$. If we let $C_2 = 0$ in equation (4.2), then $\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1$ is an exponential solution. This solution is illustrated in Figure 1 for the two cases when $C_1 = 0.8$ and $C_1 = -0.8$. For every value of t, $\mathbf{y}(t)$ is a positive multiple of the eigenvector $C_1\mathbf{v}_1$. Thus, the solution curve traces out the half-line through the origin with direction $C_1\mathbf{v}_1$. We will call that the ***half-line generated by*** $C_1\mathbf{v}_1$. This half-line coincides with the half-line generated by $\mathbf{v}_1$ if $C_1 > 0$ and with the half-line generated by $-\mathbf{v}_1$ if $C_1 < 0$. Notice that the multiplying factor is the exponential function $e^{\lambda_1 t}$. Since $\lambda_1 < 0$, this function decreases from ∞ to 0 as t varies from $-\infty$ to ∞. Thus, $\mathbf{y}(t)$ starts for $t = -\infty$ infinitely far from the origin along the half-line, and as $t \to \infty$, $\mathbf{y}(t) \to \mathbf{0}$. We will sometimes refer to these exponential solutions as ***half-line solutions***.

On the other hand, if we let $C_1 = 0$ in equation (4.2), then $\mathbf{y}(t) = C_2 e^{\lambda_2 t}\mathbf{v}_2$, giving us a second exponential solution. However, because $\lambda_2 > 0$, the direction of the motion changes. At $t = -\infty$, $\mathbf{y}(t)$ starts at the origin, and as $t \to \infty$, $\mathbf{y}(t)$ moves away from the origin along the half-line generated by $C_2\mathbf{v}_2$. This exponential, or half-line, solution is illustrated in Figure 1 for the two cases when $C_2 = 1.2$ and $C_2 = -1.2$.

In the remaining, more interesting case, both constants in the general solution (4.2) are nonzero. In this case, the general solution,

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2, \tag{4.3}$$

is a superposition of the two exponential solutions. Several solutions are illustrated in Figure 2. As $t \to \infty$, the first term in (4.3), $C_1 e^{\lambda_1 t}\mathbf{v}_1$, tends to $\mathbf{0}$, and the solution gets closer to the second term, $C_2 e^{\lambda_2 t}\mathbf{v}_2$. This means that the solution curve goes to ∞, and in the process is asymptotic to the half-line generated by $C_2\mathbf{v}_2$. On the other hand, as $t \to -\infty$, $C_2 e^{\lambda_2 t}\mathbf{v}_2 \to \mathbf{0}$, so the solution gets closer to $C_1 e^{\lambda_1 t}\mathbf{v}_1$. Geometrically this means that the solution curve goes to ∞, asymptotic to the half-line generated by $C_1\mathbf{v}_1$.

This behavior is illustrated by the following example.

EXAMPLE 4.4 ◆ The system

$$\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\mathbf{y} \tag{4.5}$$

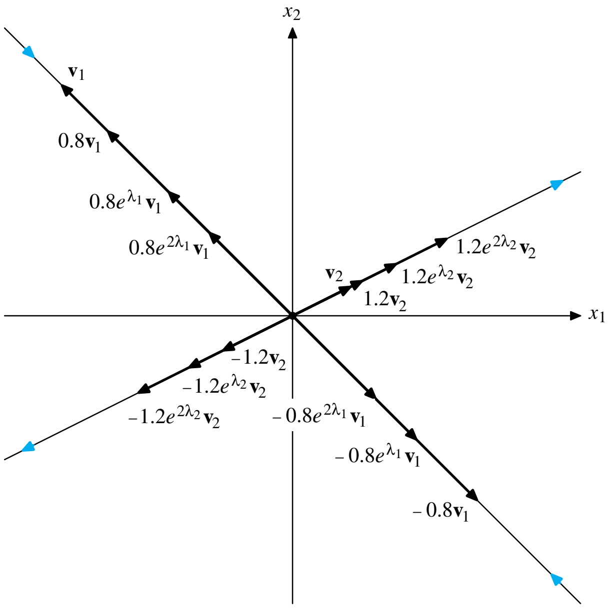


Figure 1 Exponential solutions near a saddle point. The direction of flow is indicated by the blue arrows. The black arrows indicate the position of the solution for various values of *t*.

has eigenvalues -3 and 3, with corresponding eigenvectors $(-1, 1)^T$ and $(2, 1)^T$, respectively. The eigenvalues are distinct, making

$$\mathbf{y}_1(t) = e^{-3t}\begin{pmatrix}-1\\1\end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = e^{3t}\begin{pmatrix}2\\1\end{pmatrix}$$

a fundamental set of solutions. Consequently, the general solution of system (4.5) is

$$\mathbf{y}(t) = C_1e^{-3t}\begin{pmatrix}-1\\1\end{pmatrix} + C_2e^{3t}\begin{pmatrix}2\\1\end{pmatrix}. \tag{4.6}$$

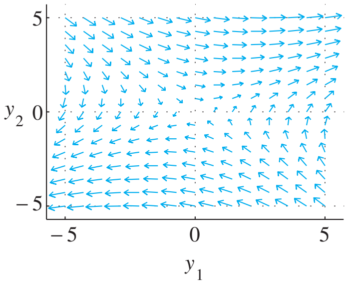


Figure 3 The arrows of the vector field indicate the presence of half-line solutions along y = –x and y = (1/2)x.

We've used our numerical solver to sketch the vector field in Figure 3 associated with system (4.5).

If $C_1 = 0$, then the solution is $y(t) = C_2e^{3t}(2, 1)^T$, which traces out half-lines emanating from the origin along the line $y = x/2$. Note how the arrows of the vector field in Figure 3 clearly indicate that these half-line solutions move away from the origin, as is expected with a growth coefficient e^{3t}. In a similar manner, if $C_2 = 0$, then the solution is $\mathbf{y}(t) = C_1e^{-3t}(-1, 1)^T$, which traces out half-line solutions decaying into the origin along the line $y = -x$. This time, the arrows in the vector field in Figure 3 indicate that these half-line solutions move toward the origin, as is expected with a decay coefficient e^{-3t}.

In Figure 4, we've used our numerical solver to draw the half-line solutions.[1] Remember, solution trajectories cannot cross one another, so the half-line solutions

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[1] Our solver draws both forward and backward solutions. So drawing half-line solutions in this case is a simple matter of drawing trajectories with initial conditions at (2, 1), (−2, −1), (−1, 1), and (1, −1).

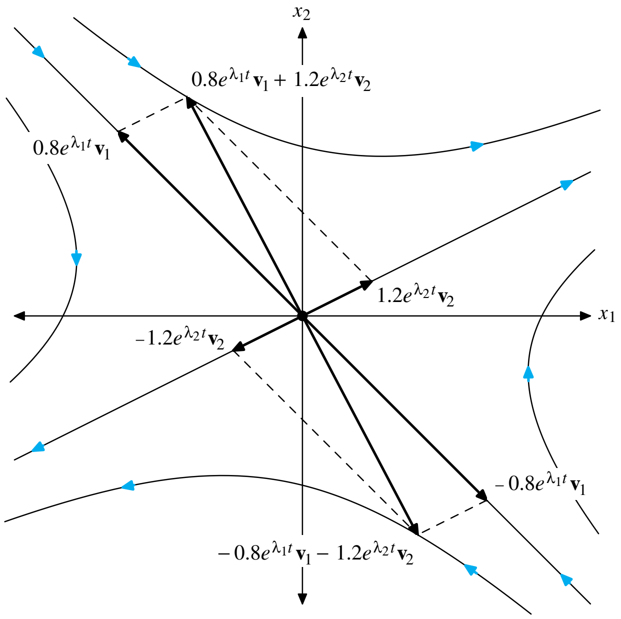

Figure 2 As $t \to \infty$, the solution $\mathbf{y}(t) = 0.8e^{\lambda_1 t}\,\mathbf{v}_1 + 1.2e^{\lambda_2 t}\,\mathbf{v}_2$ moves near $1.2e^{\lambda_2 t}\,\mathbf{v}_2$. As $t \to -\infty$, the solution moves near $0.8e^{\lambda_1 t}\,\mathbf{v}_1$

"separate" the phase plane into four distinct regions. We've plotted trajectories with initial conditions in each of these four regions. Each of these trajectories depicts the typical case where neither C_1 nor C_2 equal zero. As time moves forward, $C_1e^{-3t}(-1, 1)^T$ decays to $\mathbf{0}$, so solutions tend toward $C_2e^{3t}(2, 1)^T$, as is clearly evidenced in Figure 4. As $t \to -\infty$, $C_2e^{3t}(2, 1)^T$ approaches $\mathbf{0}$. Consequently, as we move backward in time, solutions move toward $C_1e^{-3t}(-1, 1)^T$, as is also evidenced in Figure 4.

◆

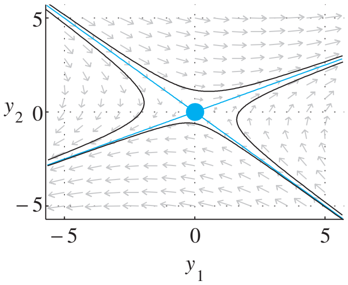

Figure 4 As $t \to \infty$, solution curves approach the half-line generated by $C_2(2,1)^T$.

As Figures 2 and 4 show, the exponential, or half-line, solutions separate the plane into four regions in which the solution curves have different behavior. For this reason, these curves (half-lines) are called ***separatrices***.

The distinguishing characteristic of a system of the type that we are considering here is the existence of the separatrices. Two of these are solution curves that approach the equilibrium point as $t \to \infty$. These are called the ***stable solutions***. Two others approach the equilibrium point as $t \to -\infty$. These are called the ***unstable solutions***. Any equilibrium point of a planar system (linear or nonlinear) that has this property is called a ***saddle point***.

Finally, notice that if the solution curves were the altitude lines on a topographic map, then the surface would have the shape of a saddle. This is the reason for the name *saddle point*.

Nodal sink

Next, suppose that both eigenvalues are negative, $\lambda_1 < \lambda_2 < 0$. Again the solution is given by (4.2). If either of the constants is zero, we get an exponential solution. However, now both eigenvalues are negative, so in both cases, the exponential solutions start infinitely large at $t = -\infty$ and converge to $\mathbf{0}$ along the appropriate half-line as $t \to \infty$. For example, the exponential solutions $\mathbf{y}(t) = 0.7e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{y}(t) = 0.5e^{\lambda_2 t}\mathbf{v}_2$ in Figure 5 decay to the origin along the half-lines generated by $0.7\mathbf{v}_1$ and $0.5\mathbf{v}_2$, respectively.

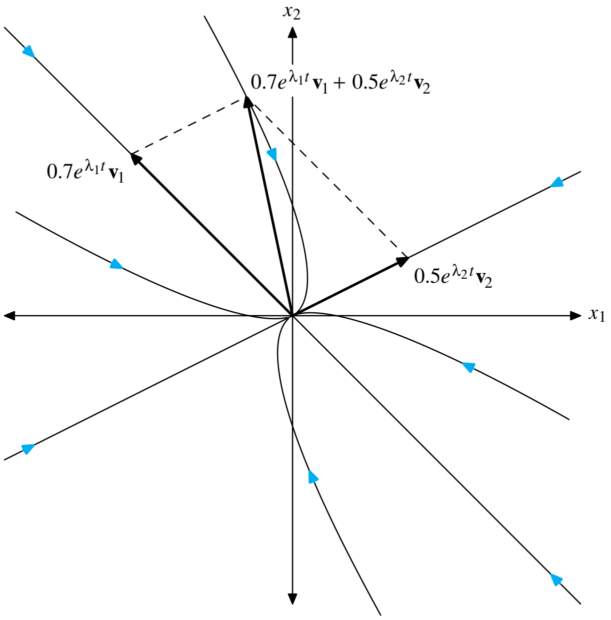

Figure 5 As $t \to \infty$, the solution $\mathbf{y}(t) = 0.7e^{\lambda_1 t}\mathbf{v}_1 + 0.5e^{\lambda_2 t}\mathbf{v}_2$ decays to the origin in a direction parallel to $0.5\mathbf{v}_2$. As $t \to -\infty$, the solution moves to infinity in a direction parallel to $0.7\mathbf{v}_1$

In the case where both C_1 and C_2 are nonzero, the general solution (4.2) is again a superposition of the exponential solutions. To examine what happens as $t \to \infty$, we rewrite (4.2) as

$$\mathbf{y}(t) = C_1e^{\lambda_1 t}\mathbf{v}_1 + C_2e^{\lambda_2 t}\mathbf{v}_2 = e^{\lambda_2 t}\left(C_1e^{(\lambda_1-\lambda_2)t}\mathbf{v}_1 + C_2\mathbf{v}_2\right). \tag{4.7}$$

Look at the two factors in the last line of (4.7). The first factor, the exponential term $e^{\lambda_2 t}$, tends toward 0 as $t \to \infty$, since $\lambda_2 < 0$. On the other hand, since $\lambda_1 - \lambda_2 < 0$, the exponential in the bracketed term also goes to 0. Thus, the bracketed factor converges to $C_2\mathbf{v}_2$. Consequently, the product, $\mathbf{y}(t)$, converges to $\mathbf{0}$, but in the process, its direction gets closer to that of $C_2\mathbf{v}_2$. This means that as $\mathbf{y}(t) \to \mathbf{0}$, the solution curve becomes tangent to the half-line generated by $C_2\mathbf{v}_2$.[2]

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[2] There is a nice way to remember this behavior. Because $\lambda_1 < \lambda_2 < 0$, as $t \to \infty$, the decay of

To examine what happens as $t \to -\infty$, we rewrite (4.2) as

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = e^{\lambda_1 t}\left(C_1\mathbf{v}_1 + C_2 e^{(\lambda_2-\lambda_1)t}\mathbf{v}_2\right).$$

Now the exponential term $e^{\lambda_1 t}$ goes to ∞, since $\lambda_1 < 0$. Since $\lambda_2 - \lambda_1 > 0$, the exponential term in the bracketed factor converges to $\mathbf{0}$. Thus, the bracketed factor converges to $C_1\mathbf{v}_1$. Therefore, as $t \to -\infty$, the product $\mathbf{y}(t)$ gets infinitely large, but in the process, its direction approaches that of $C_1\mathbf{v}_1$.[3] This behavior is shown in Figure 5 for several choices of C_1 and C_2.

A distinguishing characteristic of a planar linear system with two negative eigenvalues is that all solution curves approach the origin as $t \to \infty$ with a well-defined tangent line. For the exponential solution $e^{\lambda_1 t}\mathbf{v}_1$, the solution approaches the origin along the line generated by $\mathbf{v}_1$. All other solutions approach the origin tangent to the line generated by $\mathbf{v}_2$. Any equilibrium point for a planar system, linear or nonlinear, that has the property that all solution curves approach the equilibrium point as $t \to \infty$ with a well-defined tangent is called a ***nodal sink***. If all solution curves approach the equilibrium point as $t \to -\infty$ with a well-defined tangent, the equilibrium point is called a ***nodal source***. Thus, we see that a planar linear system with two negative eigenvalues has a nodal sink at the origin.

Let's look at an example of a nodal sink.

EXAMPLE 4.8 ◆

The system

$$\mathbf{y}' = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} \tag{4.9}$$

has eigenvalues -2 and -4, with corresponding eigenvectors $(-1, 1)^T$ and $(1, 1)^T$, respectively. The eigenvalues are distinct, making

$$\mathbf{y}_1(t) = e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

a fundamental set of solutions. Consequently, the general solution of system (4.9) is

$$\mathbf{y}(t) = C_1 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.10}$$

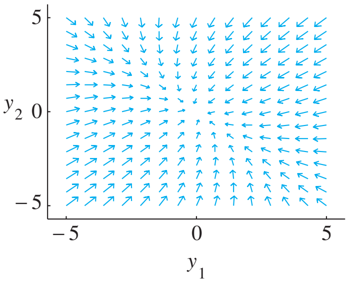

Figure 6 The arrows of the vector field indicate the presence of half-line solutions along the lines y = −x and y = x.

We've used our numerical solver to sketch the vector field shown in Figure 6 associated with system (4.10).

The exponential solution $\mathbf{y}(t) = C_1 e^{-2t}(-1, 1)^T$ traces out half-line solutions that decay to the origin along the line $y = -x$. Note how the arrows of the vector field in Figure 6 clearly indicate that these solutions move toward the origin, as is expected with the decay coefficient e^{-2t}. The second exponential solution, $\mathbf{y}(t) = C_2 e^{-4t}(1, 1)^T$, is also evident in Figure 6, where the arrows indicate half-line solutions decaying to the origin along the line $y = x$.

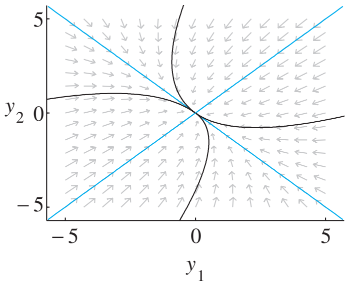

Figure 7 As $t \to \infty$, solution curves approach the origin tangent to the half-line generated by $C_1(-1,1)^T$.

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the exponential solution $\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1$ is more rapid ("faster") than that of the exponential solution, $\mathbf{y}(t) = C_2 e^{\lambda_2 t}\mathbf{v}_2$. The general solution decays to the origin along a direction eventually paralleling that of the "slower" exponential solution, $\mathbf{y}(t) = C_2 e^{\lambda_2 t}\mathbf{v}_2$.

[3] As $t \to -\infty$, the trajectory goes to infinity, eventually turning parallel to the "faster" exponential solution, $\mathbf{y}(t) = e^{\lambda_1 t}\mathbf{v}_1$.

In the phase portrait shown in Figure 7, we've drawn the half-line solutions and four solutions where neither C_1 nor C_2 equals zero. As $t \to \infty$, note how each of these latter trajectories decays to the origin in a direction parallel to $C_1 e^{-2t}(-1, 1)^T$ (the "slower" exponential solution). As $t \to -\infty$, these solutions move toward infinity in a direction eventually paralleling the vector $C_2 e^{-4t}(1, 1)^T$ (the "faster" exponential solution). ◆

Nodal source

Next, suppose both eigenvalues are positive, $0 < \lambda_1 < \lambda_2$. The argument in this case parallels that in the description of a nodal sink, only with time reversed. Let's look at an example.

EXAMPLE 4.11 ◆ The system

$$\mathbf{y}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \tag{4.12}$$

has eigenvalues 2 and 4, with corresponding eigenvectors $(-1, 1)^T$ and $(1, 1)^T$, respectively. The general solution is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.13}$$

Note that equation (4.13) is identical to equation (4.10), with $-t$ replaced with t. Consequently, it should come as no surprise when this system's phase portrait in Figure 8 duplicates the image in Figure 7, only with time reversed.

Indeed, since all solution curves approach the origin as $t \to -\infty$ with a definite tangent, the origin is a nodal source. ◆

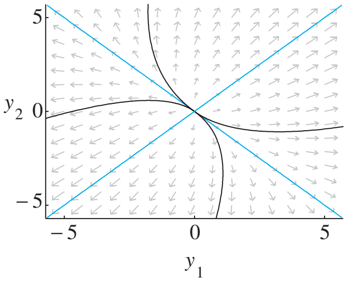

Figure 8 As $t \to \infty$, solution move to infinity in a direction parallel to $C_2(1,1)^T$.

Complex eigenvalues

The next three cases will study situations where the eigenvalues are complex. In Theorem 3.28, we saw that a complex eigenvalue $\lambda = \alpha + i\beta$ and its associated eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ lead to the general solution

$$\mathbf{y}(t) = C_1 e^{\alpha t}(\cos \beta t\, \mathbf{v}_1 - \sin \beta t\, \mathbf{v}_2) + C_2 e^{\alpha t}(\sin \beta t\, \mathbf{v}_1 + \cos \beta t\, \mathbf{v}_2). \tag{4.14}$$

Center

Suppose the eigenvalues are purely imaginary. Then $\alpha = 0$, and equation (4.14) becomes

$$\mathbf{y}(t) = C_1(\cos \beta t\, \mathbf{v}_1 - \sin \beta t\, \mathbf{v}_2) + C_2(\sin \beta t\, \mathbf{v}_1 + \cos \beta t\, \mathbf{v}_2). \tag{4.15}$$

The trigonometric functions $\cos \beta t$ and $\sin \beta t$ are both periodic with period $T = 2\pi/|\beta|$ and frequency $|\beta|$. Consequently, the vector-valued function $\mathbf{y}(t)$ has the same properties. Therefore, the solution trajectory is a *closed* curve, orbiting about the origin with period $2\pi/|\beta|$.

EXAMPLE 4.16 ◆ The system

$$\mathbf{y}' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{y}, \tag{4.17}$$

has an eigenvalue–eigenvector pair $\lambda = 2i$, $\mathbf{w} = (1, i)^T$. From (4.15), we see that the general solution is

$$\mathbf{y}(t) = C_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + C_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}. \tag{4.18}$$

We can show that the solution curves are circles in this case by showing that the components of $\mathbf{y}(t)$ satisfy $y_1^2 + y_2^2 =$ constant. We do this by showing that the derivative is zero. First,

$$(y_1^2 + y_2^2)' = 2y_1y_1' + 2y_2y_2'.$$

Now we use the system (4.17), which says that $y_1' = 2y_2$ and $y_2' = -2y_1$. Substituting, we get

$$(y_1^2 + y_2^2)' = 2y_1(2y_2) + 2y_2(-2y_1) = 0.$$

Thus the solution curves are circles centered at the origin. This is evident when we use a numerical solver to generate solution trajectories of system (4.17), as we have done in Figure 9. ◆

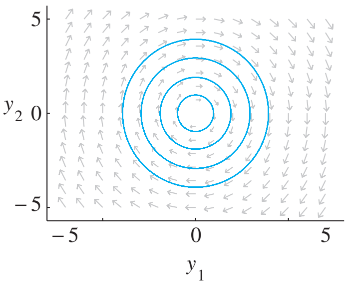

Figure 9 Circular solutions spiral about the center at the origin.

The distinguishing characteristic of this type of equilibrium point is that it is surrounded by closed solution curves. An equilibrium point for any planar system, linear or nonlinear, that has this property is called a ***center***. Thus, planar linear systems with purely imaginary eigenvalues have centers at the origin.

Not all centers have solution curves that are circles. This is not even true for linear systems. We will prove later that, for a linear center, the orbits are similar ellipses centered at the origin. For example, the system

$$\mathbf{y}' = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix} \tag{4.19}$$

has eigenvalue $2i$ and associated eigenvector $(2 + i, 1)^T$, so the equilibrium point is still a center, but the solution curves are ellipses, as shown in Figure 10.

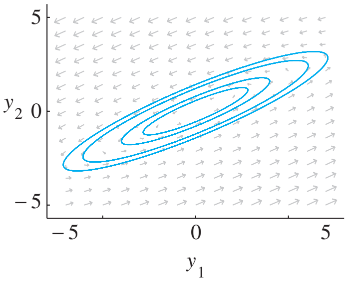

Figure 10 Elliptical solutions spiral about the center at the origin.

Spiral sink

Now suppose that the real part of the eigenvalue is negative. The solution is again provided by equation (4.14),

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{\alpha t} (\cos \beta t\, \mathbf{v}_1 - \sin \beta t\, \mathbf{v}_2) + C_2 e^{\alpha t} (\sin \beta t\, \mathbf{v}_1 + \cos \beta t\, \mathbf{v}_2) \\ &= e^{\alpha t} \left\{ C_1 (\cos \beta t\, \mathbf{v}_1 - \sin \beta t\, \mathbf{v}_2) + C_2 (\sin \beta t\, \mathbf{v}_1 + \cos \beta t\, \mathbf{v}_2) \right\}. \end{aligned} \tag{4.20}$$

The term inside the brackets in (4.20) is just what we see in (4.15). As we have seen, these terms by themselves parametrize ellipses centered at the origin. However, these are modified by the factor $e^{\alpha t}$. Since α, the real part of the complex eigenvalue, is negative, $e^{\alpha t} \to 0$ as $t \to \infty$. Thus, while the solution curve circles the origin, it is being drawn toward it at the same time, resulting in a spiral motion. The natural frequency of the spiral motion is β, so we would expect the trajectories to complete one revolution about the origin in the time period $T = 2\pi/|\beta|$.

The fact that all solutions spiral around the equilibrium point while at the same time approaching it characterizes a ***spiral sink***. The term is used for nonlinear systems as well. Thus, any linear system with complex eigenvalues having negative real parts has a spiral sink at the origin. Let's look at an example.

EXAMPLE 4.21 ◆ The system

$$\mathbf{y}' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} \tag{4.22}$$

has an eigenvalue–eigenvector pair $\lambda = -1 + 2i$, $\mathbf{w} = (2, 1 - i)^T$. Because $\alpha = \mathrm{Re}(\lambda) = -1 < 0$, we expect solutions to decay to the origin. In addition, because $\beta = \mathrm{Im}(\lambda) = 2$, the natural frequency is 2, and the time to complete one revolution is $T = 2\pi/2 = \pi$. These claims are evident in Figure 11, where we've used our numerical solver to start a trajectory at a particular initial condition, but restricted the time of computation to the interval $[0, \pi]$. ◆

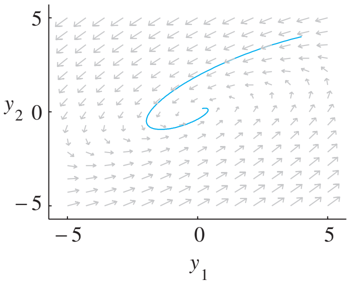

Figure 11 The solution discussed in Example 4.21.

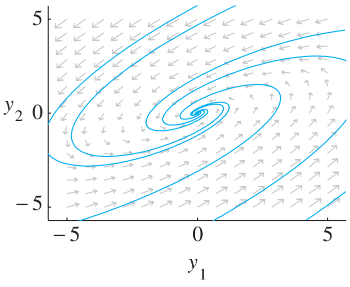

Figure 12 Solutions spiral and decay to the origin.

In the phase portrait shown in Figure 12, a number of solution trajectories swirl about the origin, decaying to $\mathbf{0}$ as $t \to \infty$. This illustrates the behavior we expect near a spiral sink.

Spiral source

Suppose the real part of the eigenvalue is positive. Again, the general solution is given by (4.20), but now $\alpha > 0$. Therefore, the amplitude of oscillation will increase as solutions spiral about the origin, since $e^{\alpha t} \to \infty$ as $t \to \infty$. This behavior characterizes a ***spiral source.*** Thus, a linear system with complex eigenvalues having a positive real part has a spiral source at the origin.

For example, the system

$$\mathbf{y}' = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \mathbf{y} \tag{4.23}$$

has eigenvalue $\lambda = 1 + i$. A phase portrait for system (4.23) appears in Figure 13. The behavior seen in Figure 13 illustrates what is expected near a ***spiral source***.

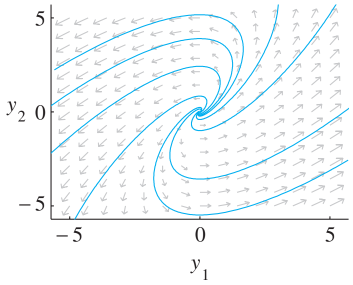

Figure 13 Solutions spiral and grow away from the origin.

The direction of rotation

If you don't have your numerical solver handy and you want a quick sketch, an immediate problem arises. The real part of the eigenvalue $\lambda = 1 + i$ is positive, indicating a spiral source, but is the motion clockwise or counterclockwise? One quick way to determine the rotation direction is simply to compute one vector in the vector field determined by the right-hand side of system (4.23). For example, determine the vector at the point (0.5, 0) with

$$\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Sketch the vector $(1, 1)^T$ at the point $(0.5, 0)$. Because the solution trajectory must be tangent to the vector $(1, 1)^T$ at the point $(0.5, 0)$, it is clear that the rotation is counterclockwise. This is shown in Figure 14.

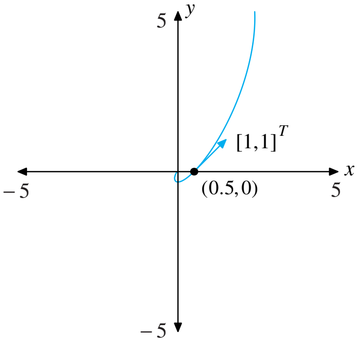

Figure 14 Often, the computation of a single vector determines the direction of rotation.

More generally, if we have a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

that has complex eigenvalues, then we can compute the vector field at $(1, 0)^T$. This gives

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}.$$

If $a_{21} > 0$, this vector points into the upper half-plane, indicating that the rotation is counterclockwise. On the other hand, if $a_{21} < 0$, this vector points into the lower half-plane, indicating that the rotation is clockwise.

The trace-determinant plane

Up to this point, our analysis of equilibrium points has not seemed to be systematic. Now we will correct that.

In Section 9.2, we showed that the characteristic polynomial of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is

$$p(\lambda) = \lambda^2 - T\lambda + D, \tag{4.24}$$

where $T = \text{tr}(A) = a_{11} + a_{22}$ and $D = \det(A) = a_{11}a_{22} - a_{21}a_{12}$. The roots of the characteristic polynomial, which are the eigenvalues of the matrix A, are determined by T and D using the quadratic formula,

$$\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}. \tag{4.25}$$

The fundamental theorem of algebra dictates that the characteristic polynomial must factor as

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2),$$

where λ_1 and λ_2 are the eigenvalues of the matrix. If we multiply out the right-hand side and collect powers of λ, the characteristic polynomial becomes

$$p(\lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2. \tag{4.26}$$

Comparing the coefficients of the powers of λ in (4.24) and (4.26), we discover the important relationships,

$$D = \det(A) = \lambda_1\lambda_2 \quad \text{and} \quad T = \operatorname{tr}(A) = \lambda_1 + \lambda_2. \tag{4.27}$$

Thus, the eigenvalues determine the trace and the determinant as well, and formulas (4.25) and (4.27) complement each other.

This duality between pairs of eigenvalues and trace-determinant pairs enables us to systematize our analysis of possible equilibrium points in terms of the trace and the determinant. Since both the trace and the determinant are real, we can use the ***trace-determinant plane*** to provide visual assistance to our efforts. This is simply a coordinate plane with coordinates T and D. (See Figures 15 and 16.)

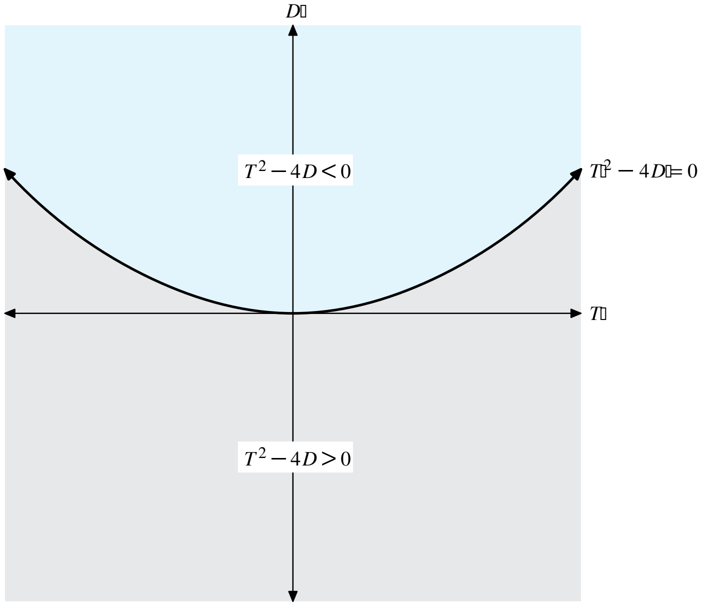

Figure 15 The trace-determinant plane.

The quadratic formula (4.25) for the eigenvalues shows that the discriminant, T^2-4D, plays a special role. If $T^2-4D>0$, there are two real eigenvalues. If $T^2-4D=0$, we have one repeated eigenvalue. Finally, if $T^2-4D<0$, the eigenvalues are complex conjugates. Clearly, the sign of the discriminant is important, so we begin by sketching the graph of $T^2-4D=0$ in the trace-determinant plane, as shown in Figure 15. The parabola $T^2-4D=0$ divides the trace-determinant plane into two separate regions.

In the region above the parabola, we have $T^2-4D<0$.[4] Therefore, matrices having trace T and determinant D, where (T, D) lies above the parabola, have complex eigenvalues. In this case, the type of the equilibrium point the system has at the origin is determined by the real part of the eigenvalue. According to equation (4.25), the real part of the eigenvalue is $T/2$. Consequently, if $T<0$, we have a spiral sink; if $T=0$, we have a center; and if $T>0$, we have a spiral source. These regions are labeled in Figure 16.

In the region below the parabola, we have $T^2-4D>0$, so the eigenvalues are real. Let's focus on the determinant $D=\lambda_1\lambda_2$. If $D<0$, the eigenvalues must have opposite signs. Hence, the equilibrium point is a saddle point. Thus, the half-plane below the T-axis corresponds to saddle points. On the other hand, if (T, D) is below

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[4] This is easily checked. Note that the point (0, 1) satisfies the inequality $T^2-4D<0$ (substitute $T=0$ and $D=1$). This indicates that the region above the parabola satisfies $T^2-4D<0$.

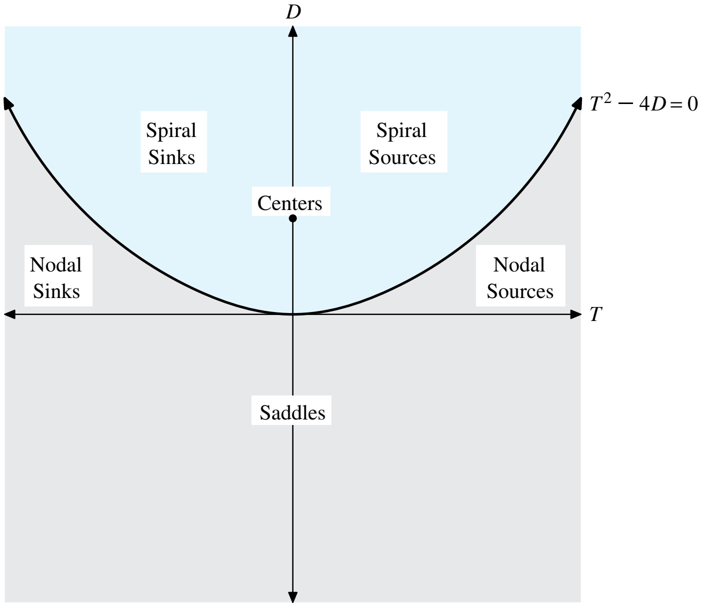

Figure 16 Classifying equilibrium points in trace-determinant plane.

the parabola but above the T-axis, then $D = \lambda_1\lambda_2 > 0$. Therefore, the eigenvalues have the same sign. The sign is determined by the trace $T = \lambda_1 + \lambda_2$. If T is positive, both eigenvalues are positive, and the equilibrium point is a nodal source. However, if T is negative, both eigenvalues are negative and the equilibrium point is a nodal sink.

All of our findings are shown in Figure 16. You will notice that we have analyzed the equilibrium points for almost every point in the trace-determinant plane. Let's look specifically at the following five types:

- **Saddle point.** A has two real eigenvalues, one positive and one negative.
- **Nodal sink.** A has two real and negative eigenvalues.
- **Nodal source.** A has two real and positive eigenvalues.
- **Spiral sink.** A has two complex conjugate eigenvalues with a negative real part.
- **Spiral source.** A has two complex conjugate eigenvalues with a positive real part.

Each of these types corresponds to a large open subset of the trace-determinant plane (Figure 16). For that reason, we will call each of these five types ***generic***. While almost all points in the trace-determinant plane correspond to one of the five generic types, there are exceptions. All of these exceptional types of equilibrium points are called ***nongeneric***.

Most important among the nongeneric equilibrium points is the center. This fact will be important when we get to the analysis of nonlinear systems. It remains to analyze the nongeneric cases that lie on the T-axis and on the parabola itself. We will examine these scenarios in the exercises.

Let's look at some examples.

EXAMPLE 4.28 ◆ Consider the system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 4 & -3 \\ 15 & -8 \end{pmatrix}.$$

The trace is $T = -4$ and the determinant is $D = 13$. Hence, $T^2 - 4D = -36 < 0$. Consequently, this puts us in the region of the trace-determinant plane inhabited by spiral sinks. This is exactly the behavior produced by our numerical solver in Figure 17. ◆

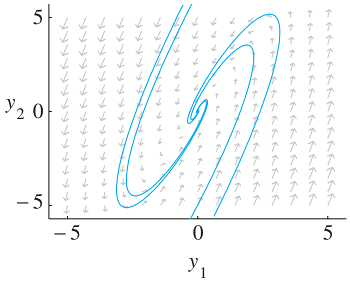

Figure 17 The spiral sink in Example 4.28.

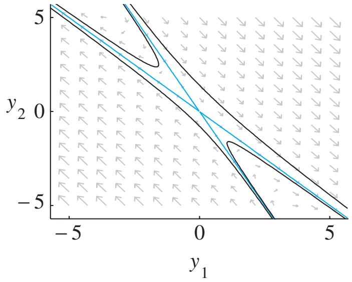

Figure 18 The saddle point in Example 4.29.

EXAMPLE 4.29 ◆ Consider the system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 8 & 5 \\ -10 & -7 \end{pmatrix}.$$

The trace is $T = 1$ and the determinant is $D = -6$. This places us firmly in the region below the T-axis. Consequently, the equilibrium point at the origin is a saddle point. This is exactly the behavior produced by our numerical solver in Figure 18. ◆

EXAMPLE 4.30 ◆ Consider the system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}.$$

The trace is $T = -3$ and the determinant is $D = 2$. Moreover, $T^2 - 4D = 1$. This places us in the region of nodal sinks. This analysis is supported by the phase portrait produced by our numerical solver in Figure 19. ◆

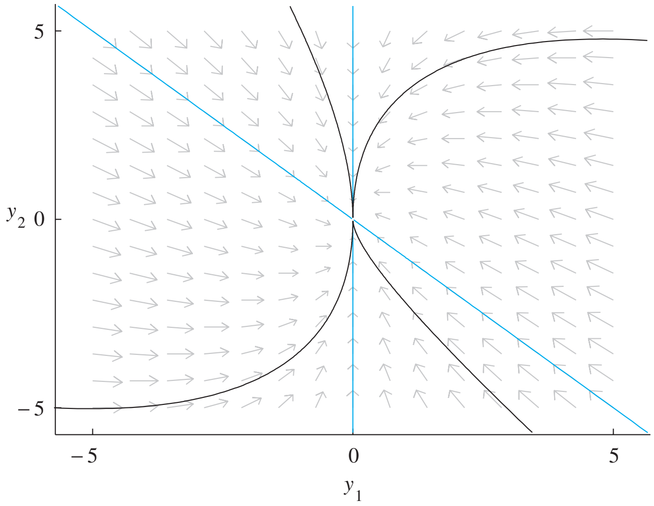

Figure 19 The nodal sink in Example 4.30.

9.5 Higher Dimensional Systems

In Sections 9.1, 9.2, and 9.3, we completely resolved the questions of solving systems of dimension 2. Much of what we did applies to higher dimensional systems as well. For example, Theorem 2.3 is still valid and will be one of our most important tools. To remind you, it says that if λ is an eigenvalue of a matrix A and $\mathbf{v}$ is an associated eigenvector, then $\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$ is the solution to $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{v}$.

We continue to have the problem of showing that solutions are linearly independent. Proposition 3.3 can be generalized to help us.

PROPOSITION 5.1 Suppose $\lambda_1, \dots,$ and λ_k are distinct eigenvalues for an $n \times n$ matrix A. If $\mathbf{v}_i \neq \mathbf{0}$ is an eigenvector for λ_i, $1 \leq i \leq k$, then $\mathbf{v}_1, \dots,$ and $\mathbf{v}_k$ are linearly independent.

The idea for the proof of Proposition 5.1 is illustrated in the proof of Proposition 3.3.

Real, distinct eigenvalues

We can link Theorem 2.3 and Proposition 5.1 to solve any system with real, distinct eigenvalues.

THEOREM 5.2 Suppose the $n \times n$ matrix A has n distinct eigenvalues $\lambda_1, \dots,$ and λ_n. Suppose that for $1 \leq i \leq n$, $\mathbf{v}_i$ is a nonzero eigenvector associated with λ_i. Then the n exponential solutions $\mathbf{y}_i(t) = e^{tA}\mathbf{v}_i = e^{\lambda_i t}\mathbf{v}_i$ form a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$.

Proof We only need to show that the exponential solutions are linearly independent. Notice that $\mathbf{y}_i(0) = \mathbf{v}_i$, for $1 \leq i \leq n$. By Proposition 5.1, the eigenvectors are linearly independent. Hence, our solutions are as well.

The main application of Theorem 5.2 is when the eigenvalues are real. In this case, the exponential solutions are also real. Hence, the general solution has the form

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + C_n e^{\lambda_n t}\mathbf{v}_n, \tag{5.3}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

EXAMPLE 5.4 ◆ Find the general solution to the three-dimensional system

$$\mathbf{y}' = A\mathbf{y}, \quad \text{where} \quad A = \begin{pmatrix} -9 & -3 & -7 \\ 3 & 1 & 3 \\ 11 & 3 & 9 \end{pmatrix}.$$

The eigenvalues are the solutions to the characteristic equation

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\begin{pmatrix} -9-\lambda & -3 & -7 \\ 3 & 1-\lambda & 3 \\ 11 & 3 & 9-\lambda \end{pmatrix} \\ &= -\lambda^3 + \lambda^2 + 4\lambda - 4. \end{aligned}$$

The roots can be found by checking whether the factors of the constant term, 4, are roots. It is discovered that the roots are $\lambda = 1, 2$, and -2.

Now we find the associated eigenvectors. When $\lambda = 1$, the matrix $A - \lambda I$ becomes

$$\begin{pmatrix} -9-\lambda & -3 & -7 \\ 3 & 1-\lambda & 3 \\ 11 & 3 & 9-\lambda \end{pmatrix} = \begin{pmatrix} -10 & -3 & -7 \\ 3 & 0 & 3 \\ 11 & 3 & 8 \end{pmatrix}.$$

The eigenvectors are vectors in the nullspace of this matrix. Using the methods we discussed in Chapter 6, we find that the nullspace is generated by the vector

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

By a similar procedure, an eigenvector corresponding to $\lambda_2 = 2$ is

$$\mathbf{v}_2 = \begin{pmatrix} 4 \\ -3 \\ -5 \end{pmatrix},$$

and an eigenvector corresponding to $\lambda_3 = -2$ is

$$\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

The general solution is

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + C_3 e^{\lambda_3 t}\mathbf{v}_3 \\ &= C_1 e^{t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 4 \\ -3 \\ -5 \end{pmatrix} + C_3 e^{-2t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

◆

Complex eigenvalues

Theorem 5.2 applies if some or all of the eigenvalues are complex, as long as they are distinct. However, as we discovered in Section 9.3, the exponential solutions corresponding to complex eigenvalues are complex valued. Here we will show what to do to get real-valued solutions.

We are looking at a system in which the matrix A has real entries. Hence the characteristic polynomial has real coefficients. Since the eigenvalues are the roots of this polynomial, they are either real or they appear in complex conjugate pairs. Suppose that λ and $\overline{\lambda}$ are a complex conjugate pair of eigenvalues of A. Then, as we discovered in Section 9.3, we have complex conjugate eigenvectors $\mathbf{w}$ and $\overline{\mathbf{w}}$ associated with λ and $\overline{\lambda}$, respectively. These lead to two complex conjugate solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} \quad \text{and} \quad \overline{\mathbf{z}}(t) = e^{\overline{\lambda}t}\,\overline{\mathbf{w}}.$$

According to Proposition 3.25, the real and imaginary parts of $\mathbf{z}$ are also solutions. These solutions can be used instead of the complex solutions in the fundamental set of solutions. We proved in Proposition 3.25 that if the complex conjugate pair of solutions $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent, then the same is true for the real and imaginary parts $\mathbf{x} = \operatorname{Re}\mathbf{z}$ and $\mathbf{y} = \operatorname{Im}\mathbf{z}$.

EXAMPLE 5.5 ◆ Find a fundamental set of solutions for the system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 14 & 8 & -19 \\ -40 & -25 & 52 \\ -5 & -4 & 6 \end{pmatrix}.$$

The first step is to find the eigenvalues. For matrices as complicated as these, it is best to use a computer. Our computer tells us that the eigenvalues are -1, $-2+3i$, and $-2-3i$. Thus, we have one real eigenvalue and a pair of complex conjugate eigenvalues.

For the real eigenvalue, $\lambda = -1$, we use the method in Theorem 2.3. We look for an eigenvector, which is a vector in the nullspace of

$$A - \lambda I = A + I = \begin{pmatrix} 15 & 8 & -19 \\ -40 & -24 & 52 \\ -5 & -4 & 7 \end{pmatrix}.$$

Again, our computer was used to find that a reasonable choice of an eigenvector is $\mathbf{v}_1 = (2, 1, 2)^T$. Our first solution is

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

For the complex conjugate pair, we follow the technique just described. We look for an eigenvector associated with the eigenvalue $\lambda = -2+3i$. Our computer tells us that $\mathbf{w} = (i, 2-2i, 1)^T$ is a reasonable choice. Thus, our complex-valued solutions are

$$\mathbf{z}(t) = e^{tA}\mathbf{w} = e^{(-2+3i)t}\begin{pmatrix} i \\ 2-2i \\ 1 \end{pmatrix} \quad \text{and} \quad \overline{\mathbf{z}}(t) = e^{tA}\overline{\mathbf{w}} = e^{(-2-3i)t}\begin{pmatrix} -i \\ 2+2i \\ 1 \end{pmatrix}.$$

Using Euler's formula, we can find the real and imaginary parts of $\mathbf{z}$.

$$\begin{aligned} \mathbf{z}(t) &= e^{-2t}e^{3it}\begin{pmatrix} i \\ 2-2i \\ 1 \end{pmatrix} \\ &= e^{-2t}\left(\cos 3t + i \sin 3t\right)\left(\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right) \\ &= e^{-2t}\left\{\begin{pmatrix} -\sin 3t \\ 2\cos 3t + 2\sin 3t \\ \cos 3t \end{pmatrix} + i\begin{pmatrix} \cos 3t \\ 2\sin 3t - 2\cos 3t \\ \sin 3t \end{pmatrix}\right\} \end{aligned}$$

We know that the real and imaginary parts of $\mathbf{z}$ are also solutions, so our two new solutions are

$$\begin{aligned} \mathbf{x}_2(t) &= e^{-2t}\begin{pmatrix} -\sin 3t \\ 2\cos 3t + 2\sin 3t \\ \cos 3t \end{pmatrix} \quad \text{and} \\ \mathbf{x}_3(t) &= e^{-2t}\begin{pmatrix} \cos 3t \\ 2\sin 3t - 2\cos 3t \\ \sin 3t \end{pmatrix}. \end{aligned}$$

Since the matrix A has three different eigenvalues, which give rise to these three solutions, we can be sure that $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent. Consequently, they form a fundamental set of solutions. ◆

The method used in the example works in general to find real solutions from complex conjugate pairs of eigenvalues. When there is a total of n distinct eigenvalues, we get a fundamental set of solutions in this way.

Repeated eigenvalues

If the eigenvalues are distinct, we now know how to find a fundamental set of real solutions. We have seen in Section 9.3 that repeated eigenvalues need special consideration, even in dimension 2. We expect that this situation will only get worse in higher dimensions. However, in some cases, the problem does not arise, as the next example shows.

EXAMPLE 5.6 ◆ Find a fundamental set of solutions for the system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 2 & 2 & -4 \\ 2 & -1 & -2 \\ 4 & 2 & -6 \end{pmatrix}.$$

Using our computer, we find that the characteristic polynomial of A is

$$p(\lambda) = -\lambda^3 - 5\lambda^2 - 8\lambda - 4 = -(\lambda + 1)(\lambda + 2)^2.$$

Hence, the eigenvalues are -1 and -2, and -2 is a repeated eigenvalue. For the eigenvalue -1, we find the eigenvector $\mathbf{v}_1 = (2, 1, 2)^T$. Hence, we have the solution

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

For the eigenvalue -2, we find that the eigenspace (the nullspace of $A + 2I$) has dimension 2. A basis is $\mathbf{v}_2 = (1, -2, 0)^T$ and $\mathbf{v}_3 = (1, 0, 1)^T$. For the eigenvalue $\lambda = -2$ and each of these eigenvectors, we get a solution,

$$\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{-2t}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3(t) = e^{tA}\mathbf{v}_3 = e^{-2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

At $t = 0$, we have

$$[\mathbf{x}_1(0), \mathbf{x}_2(0), \mathbf{x}_3(0)] = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Since the determinant of this matrix is -1, the three vectors are linearly independent. Consequently, the functions $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent and form a fundamental set of solutions. ◆

Example 5.6 shows that repeated eigenvalues do not necessarily cause trouble. Example 3.42 of Section 9.3, on the other hand, is a case where the repeated eigenvalue needed special handling. Here is another.

EXAMPLE 5.7 ◆ Find a fundamental set of solutions for the system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & -3 & -3 \end{pmatrix}.$$

Again, we find that the characteristic polynomial of A is

$$p(\lambda) = -\lambda^3 - 5\lambda^2 - 8\lambda - 4 = -(\lambda + 1)(\lambda + 2)^2.$$

Hence, the eigenvalues are -1 and -2, and -2 is a repeated eigenvalue. The eigenvalue -1 has $\mathbf{v}_1 = (1, 1, -2)^T$ as an eigenvector, so

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

is a solution.

This time, the eigenspace for the eigenvalue -2 has dimension 1, and it is spanned by $\mathbf{v}_2 = (1, 0, -1)^T$. Thus, we have the solution

$$\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{-2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

but we can find no more exponential solutions.

If you recall, we have never fully used Proposition 1.26 in Section 9.1. Part (2) of Proposition 1.26 says that if $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = e^{\lambda t}\left(\mathbf{v} + t[A - \lambda I]\mathbf{v}\right). \tag{5.8}$$

Because the eigenvalue -2 is repeated, let's try this with $\lambda = -2$. We compute that

$$[A + 2I]^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ -1 & -3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ -1 & -3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 9 & 0 \\ 0 & -8 & 0 \end{pmatrix}.$$

The nullspace of this matrix has dimension 2, and it has a basis $(1, 0, 0)^T$ and $(0, 0, 1)^T$. However, we have already computed the solution $\mathbf{x}_2$ using the initial value $\mathbf{v}_2 = (1, 0, -1)^T$. Notice that $\mathbf{v}_2$ is in the nullspace of $[A+2I]^2$ since $[A+2I]\mathbf{v}_2 = \mathbf{0}$. We want to choose a basis for the nullspace of $[A + 2I]^2$ which includes $\mathbf{v}_2$. We need a second vector in the nullspace of $[A + 2I]^2$ which is not a multiple of $\mathbf{v}_2$. There are lots of choices, but let's pick $\mathbf{v}_3 = (1, 0, 0)^T$. Then, using (5.8), our third solution is

$$\begin{aligned}
\mathbf{x}_3(t) &= e^{tA}\mathbf{v}_3 = e^{-2t}\left(\mathbf{v}_3 + t[A + 2I]\mathbf{v}_3\right) \\
&= e^{-2t}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ -1 & -3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \\
&= e^{-2t}\begin{pmatrix} 1+t \\ 0 \\ -t \end{pmatrix}
\end{aligned}$$

Since the vectors $\mathbf{x}_1(0) = \mathbf{v}_1$, $\mathbf{x}_2(0) = \mathbf{v}_2$, $\mathbf{x}_3(0) = \mathbf{v}_3$ are linearly independent, we have a fundamental set of solutions. ◆

Multiplicities

The resolution of Example 5.7 depended on Proposition 1.26. It will resolve all cases of higher multiplicity. However, we need to develop some new ideas.

Suppose A is an $n \times n$ matrix with real entries. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be a list of the distinct eigenvalues of A. (For example, in Examples 5.6 and 5.7, we would have $\lambda_1 = -1$ and $\lambda_2 = -2$.) In general, the characteristic polynomial of A factors into

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2}\cdots(\lambda - \lambda_k)^{q_k}.$$

The powers of the factors are at least 1 and satisfy $q_1 + q_2 + \cdots + q_k = n$. In the previous two examples, $q_1 = 1$ and $q_2 = 2$. We define the ***algebraic multiplicity*** of

λ_j to be q_j. On the other hand, the ***geometric multiplicity*** of λ_j is d_j, the dimension of the eigenspace of λ_j.

In both Examples 5.6 and 5.7, the characteristic polynomial is $-(\lambda+1)(\lambda+2)^2$. In Example 5.6, we have $d_1 = 1 = q_1$ and $d_2 = 2 = q_2$. However, in Example 5.7, we have $d_1 = 1 = q_1$ and $d_2 = 1 < q_2$. It is always true that $1 \le d_j \le q_j$. The method we used in Example 5.6 will enable us to find d_j independent solutions corresponding to the eigenvalue λ_j. It is only necessary to choose a basis for the eigenspace and use the corresponding exponential solutions. If $d_j = q_j$ for all j, then we can find a total of

$$d_1 + d_2 + \cdots + d_k = q_1 + q_2 + \cdots + q_k = n$$

solutions in this way. We would then have a fundamental set of solutions.

Consequently, we do not really need distinct eigenvalues. We can find a fundamental set of solutions provided that the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity. We will next learn how to proceed when the two multiplicities are unequal.

As Example 5.7 indicates, when eigenvalues have algebraic multiplicity greater than 1, we can compute extra solutions by looking for vectors in the nullspace of $[A-\lambda I]^p$ for $p > 1$. If λ is an eigenvector of A, and $[A-\lambda I]^p\mathbf{v} = 0$ for some integer $p \ge 1$, we will call $\mathbf{v}$ a ***generalized eigenvector***. In fact, generalized eigenvectors provide all of the solutions we need because of the following result from linear algebra.

THEOREM 5.9 Suppose λ is an eigenvector of A with algebraic multiplicity q. Then there is an integer $p \le q$ such that the dimension of the nullspace of $[A - \lambda I]^p$ is equal to q.

We now have a general procedure for finding a fundamental set of solutions for a homogeneous system $\mathbf{x}' = A\mathbf{x}$. We start by finding the eigenvalues of A and their algebraic multiplities. For each eigenvalue λ with algebraic multiplicity q find q linearly independent solutions as follows:

1. Find the smallest integer p such that the nullspace of $[A - \lambda I]^p$ has dimension q.
2. Find a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ of the nullspace of $[A - \lambda I]^p$.
3. For each $\mathbf{v}_j$, $1 \le j \le q$, we have the solution

$$\begin{aligned}\mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\ &= e^{\lambda t}\left(\mathbf{v}_j + t[A-\lambda I]\mathbf{v}_j + \cdots + \frac{t^{p-1}}{(p-1)!}[A-\lambda I]^{p-1}\mathbf{v}_j\right).\end{aligned}$$

The foregoing procedure completely solves the problem of computing a fundamental set of solutions. It is only necessary to carry out this procedure for each eigenvalue. For the eigenvalue λ_j with algebraic multiplicity q_j, we can find q_j independent solutions. That gives us $q_1 + q_2 + \cdots + q_k = n$ solutions overall. We state without proof that if the solutions chosen for each eigenvalue are linearly independent, then the n solutions will also be linearly independent.

For future reference, notice that if λ is real with algebraic multiplicity q and $\mathbf{x}_j$ is one of the solutions associated with λ using the three-step procedure in Theorem 5.9, then every component of $\mathbf{x}_j(t)$ is the product of $e^{\lambda t}$ and a polynomial of degree $p < q$.

The three-step procedure works for complex eigenvalues as well as real. However, there is a better way to proceed. Suppose $\lambda = \alpha + i\beta$ is a complex eigenvalue of algebraic multiplicity q. Then the same is true for the complex conjugate $\overline{\lambda} = \alpha - i\beta$. The solutions corresponding to $\overline{\lambda}$ are complex conjugates of solutions corresponding to λ.

1. Find the smallest integer p such that the nullspace of $[A - \lambda I]^p$ has dimension q.
2. Find a basis $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_q\}$ of the nullspace of $[A - \lambda I]^p$.
3. For each $\mathbf{w}_j$, $1 \le j \le q$, we have the solution
$$\begin{aligned}\mathbf{z}_j(t) &= e^{tA}\mathbf{w}_j \\ &= e^{\lambda t}\left(\mathbf{w}_j + t[A - \lambda I]\mathbf{w}_j + \cdots + \frac{t^{p-1}}{(p-1)!}[A - \lambda I]^{p-1}\mathbf{w}_j\right).\end{aligned}$$
4. For $1 \le j \le q$, set $\mathbf{x}_j(t) = \operatorname{Re}\mathbf{z}_j(t)$ and $\mathbf{y}_j(t) = \operatorname{Im}\mathbf{z}_j(t)$.

This will yield $2q$ real solutions for the conjugate pair of eigenvalues λ and $\overline{\lambda}$.

If $\lambda = \alpha + i\beta$ is complex with algebraic multiplicity q, and $\mathbf{x}_j$ and $\mathbf{y}_j$ are a pair of real solutions associated with λ using the four-step procedure just given, then every component $\mathbf{x}_j(t)$ or $\mathbf{y}_j(t)$ has the form

$$e^{\alpha t}\{P(t)\cos\beta t + Q(t)\sin\beta t\},$$

where P and Q are polynomials of degree $p < q$.

We should work out a few examples to see how the procedure works.

EXAMPLE 5.10 ◆ Find a fundamental set of solutions to the system

$$\mathbf{y}' = A\mathbf{y}, \quad \text{where} \quad A = \begin{pmatrix} -1 & -2 & 1 \\ 0 & -4 & 3 \\ 0 & -6 & 5 \end{pmatrix}.$$

The characteristic equation is

$$-\lambda^3 + 3\lambda + 2 = -(\lambda + 1)^2(\lambda - 2) = 0,$$

so the eigenvalues are -1 and 2, with -1 having algebraic multiplicity 2.

The eigenspace for the eigenvalue $\lambda = 2$ is spanned by

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Hence,

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{2t}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

is our first solution.

For the eigenvalue $\lambda = -1$, we have

$$A - \lambda I = A + I = \begin{pmatrix} 0 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{pmatrix} \quad \text{and} \quad [A+I]^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -9 & 9 \\ 0 & -18 & 18 \end{pmatrix}.$$

The eigenspace corresponding to $\lambda = -1$ is the nullspace of $A + I$, and it has dimension 1. Thus, the eigenvalue $\lambda = -1$ has geometric multiplicity 1. The nullspace of $[A + I]^2$ has dimension 2, which is the algebraic multiplicity of $\lambda = -1$. Therefore, we need to compute $e^{tA}\mathbf{v}$ for a basis of the nullspace of $[A + I]^2$. Notice that the nullspace of $A + I$ is a subset of the nullspace of $[A + I]^2$, so an eigenvector is a generalized eigenvector. It is usually a good idea to choose an eigenvector as part of our basis of the nullspace of $[A + I]^2$, since the solution corresponding to an eigenvector is so easy to compute. Hence, we select $\mathbf{v}_2 = (1, 0, 0)^T$. Since $\mathbf{v}_2$ is an eigenvector, the corresponding solution is

$$\mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^{-t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We need our third vector $\mathbf{v}_3$ to be in the nullspace of $[A + I]^2$, but linearly independent of $\mathbf{v}_2$. Perhaps the simplest example is $\mathbf{v}_3 = (0, 1, 1)^T$. The corresponding solution is

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA}\mathbf{v}_3 \\ &= e^{-t}\left(\mathbf{v}_3 + t[A+I]\mathbf{v}_3\right) \\ &= e^{-t}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right) \\ &= e^{-t}\begin{pmatrix} -t \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

◆

For $n \times n$ matrices with $n \leq 3$, it is possible to do all of the needed computations by hand. For larger matrices, the work gets tedious, and it does not increase our understanding. For the larger examples that follow, we encourage you to use a computer. The computations can be made quickly, and the concepts can be easily explored.

The next example will show some different features.

EXAMPLE 5.11 ◆ Find a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 7 & 5 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 12 & 10 & -5 & 4 \\ -4 & -4 & 2 & -1 \end{pmatrix}.$$

Using a computer, we find that the characteristic polynomial is

$$\lambda^4 - 2\lambda^3 + 2\lambda - 1 = (\lambda + 1)(\lambda - 1)^3.$$

Hence, the eigenvalues are $\lambda = -1$, which has algebraic multiplicity 1, and $\lambda = 1$, which has algebraic multiplicity 3.

A computer also tells us that the eigenspace for $\lambda = -1$ is generated by the vector $\mathbf{v}_1 = (1, 0, 2, -1)^T$. Since this is an eigenvector, the corresponding solution is

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}.$$

Again, a computer tells us that the nullspace of

$$A - I = \begin{pmatrix} 6 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 12 & 10 & -6 & 4 \\ -4 & -4 & 2 & -2 \end{pmatrix}$$

has dimension 2. Thus, the geometric multiplicity of $\lambda = 1$ is 2. Our computer tells us that the eigenspace is spanned by $\mathbf{v}_2 = (1, 0, 2, 0)^T$ and $\mathbf{v}_3 = (1, -2, 0, 2)^T$. Since $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors, the corresponding solutions are easily computed. They are

$$\mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^{t}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_3(t) = e^{tA}\mathbf{v}_3 = e^{t}\begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}.$$

Now we compute that the nullspace of

$$[A - I]^2 = \begin{pmatrix} -8 & -8 & 4 & -4 \\ 0 & 0 & 0 & 0 \\ -16 & -16 & 8 & -8 \\ 8 & 8 & -4 & 4 \end{pmatrix}$$

has dimension 3. Hence, we can find a third solution associated to $\lambda = 1$ by finding a vector $\mathbf{v}_4$ in the nullspace of $[A - I]^2$, which is linearly independent of $\mathbf{v}_2$ and $\mathbf{v}_3$. We will choose $\mathbf{v}_4 = (0, 0, 1, 1)^T$ because it has lots of zero entries. It is left to you to check that $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent. Since $[A - I]^2\mathbf{v}_4 = \mathbf{0}$, the corresponding solution is

$$\begin{aligned} \mathbf{y}_4(t) &= e^{tA}\mathbf{v}_4 = e^{t}\left(\mathbf{v}_4 + t[A - I]\mathbf{v}_4\right) \\ &= e^{t}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix}\right\} = e^{t}\begin{pmatrix} -t \\ 0 \\ 1 - 2t \\ 1 \end{pmatrix}. \end{aligned}$$

Since we chose $\mathbf{v}_4$ to be independent of $\mathbf{v}_2$ and $\mathbf{v}_3$, our solutions are independent and we have a fundamental set. ◆

We need an example involving complex eigenvalues.

EXAMPLE 5.12 ◆ Find a fundamental system of solutions for the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 6 & 6 & -3 & 2 \\ -4 & -4 & 2 & 0 \\ 8 & 7 & -4 & 4 \\ 1 & 0 & -1 & -2 \end{pmatrix}.$$

Using a computer, we find that the eigenvalues of A are $-1 \pm i$, each with algebraic multiplicity 2. We compute that

$$A - (-1+i)I = \begin{pmatrix} 7-i & 6 & -3 & 2 \\ -4 & -3-i & 2 & 0 \\ 8 & 7 & -3-i & 4 \\ 1 & 0 & -1 & -1-i \end{pmatrix}.$$

This matrix has a nullspace of dimension 1, so the eigenvalue $\lambda = -1 + i$ has geometric multiplicity 1. Our computer tells us that $\mathbf{w}_1 = (1+i, 0, 2+2i, -1)^T$ is an eigenvector. The corresponding complex-valued solution is

$$\mathbf{z}_1(t) = e^{tA}\mathbf{w}_1 = e^{(-1+i)t}\mathbf{w}_1 = e^{(-1+i)t} \begin{pmatrix} 1+i \\ 0 \\ 2+2i \\ -1 \end{pmatrix}.$$

To find another complex-valued solution, we compute

$$[A - (-1+i)I]^2 = \begin{pmatrix} 2-14i & 3-12i & -2+6i & -4i \\ 8i & -2+6i & -4i & 0 \\ 8-16i & 6-14i & -6+6i & -8i \\ -2-2i & -1 & 1+2i & -2+2i \end{pmatrix}.$$

We look for a vector $\mathbf{w}_2$ in this nullspace that is not an eigenvector and therefore not a multiple of $\mathbf{w}_1$. One choice is $\mathbf{w}_2 = (1+2i, -2-2i, 0, 2)^T$. Since

$$[A - (-1+i)I]^2\mathbf{w}_2 = \mathbf{0},$$

the corresponding solution is

$$\begin{aligned} \mathbf{z}_2(t) = e^{tA}\mathbf{w}_2 &= e^{(-1+i)t}\left(\mathbf{w}_2 + t[A - (-1+i)I]\mathbf{w}_2\right) \\ &= e^{(-1+i)t} \begin{pmatrix} (1+t) + i(2+t) \\ -2-2i \\ (2+2i)t \\ 2-t \end{pmatrix}. \end{aligned}$$

Corresponding to the complex conjugate eigenvalue $-1 - i$, we have the conjugate generalized eigenvectors $\overline{\mathbf{w}}_1$ and $\overline{\mathbf{w}}_2$ and the corresponding conjugate solutions $\overline{z}_1$ and $\overline{z}_2$. These are the required four solutions. To find real solutions, we use the real and imaginary parts of the complex solutions. If $\mathbf{z}_1 = \mathbf{x}_1 + i\mathbf{y}_1$ and $\mathbf{z}_2 = \mathbf{x}_2 + i\mathbf{y}_2$, then $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$, and $\mathbf{y}_2$ are the four needed real solutions. ◆