Math 211

Review for the

First Exam

September 28, 2003

Themes of the Course

- Modeling.
  - One dimensional motion, mixing problems, population growth, and personal finance.
- Exact solution methods.
  - Substitution, separable equations , and linear equations.
- Numerical solution methods.
  - Euler's method, Runge-Kutta methods, dfield6.
- Qualitative analysis.
  - Existence and uniqueness, continuity in initial conditions, sensitivity to initial conditions, and autonomous equations.

Separable Equations

$$\frac{dy}{dt} = g(y)h(t)$$

The three step solution process:

1. Separate the variables. $\dfrac{dy}{g(y)} = h(t)\,dt$ if $g(y) \neq 0$.
2. Integrate both sides. $\displaystyle\int \frac{dy}{g(y)} = \int h(t)\,dt$
3. Solve for $y(t)$.

Linear Equations — Variation of Parameters

To solve $x' = a(t)x + f(t)$:

1. Solve the homogeneous equation $x_0' = ax_0$.
2. Find v such that $x = vx_0$ is a solution by substituting into the equation.
3. Write down the general solution, $x(t) = v(t)x_0(t)$.

Linear Equations — Integrating Factor

To solve $x' = a(t)x + f(t)$:

1. Rewrite as $x' - ax = f$.
2. Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\,dt}.$$

   Equation becomes $[ux]' = ux' - aux = uf$.

3. Integrate: $u(t)x(t) = \displaystyle\int u(t)f(t)\,dt + C$.
4. Solve for $x(t)$.

Existence Theorem

Theorem: Suppose the function $f(t, y)$ is defined and continuous in the rectangle R in the ty-plane. Then given any point $(t_0, y_0) \in R$, the initial value problem

$$y' = f(t, y) \quad \text{with} \quad y(t_0) = y_0$$

has a solution $y(t)$ defined in an interval containing t_0. Furthermore the solution will be defined at least until the solution curve $t \to (t, y(t))$ leaves the rectangle R.

Uniqueness Theorem

Theorem: Suppose the function $f(t, y)$ and its partial derivative $\partial f/\partial y$ are continuous in the rectangle R in the ty-plane. Suppose that $(t_0, x_0) \in R$. Suppose that

$$x' = f(t, x) \quad \text{and} \quad y' = f(t, y),$$

and that

$$x(t_0) = y(t_0) = x_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$x(t) = y(t).$$

Autonomous Equation $y' = f(y)$

1. Graph $y \to f(y)$.
2. Find the equilibrium points where $f(y) = 0$.
3. Determine the behavior between eq. pts.
4. Analyze the equilibrium points.
5. Transfer the phase line to ty-space.
6. Plot the equilibrium solutions.
7. Plot other solutions approximately.

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