Math 211

Review for the

Second Exam

November 9, 2003

Method of Solution for $A\mathbf{x} = \mathbf{b}$

There are four steps:

1. Use the augmented matrix $M = [A,\ \mathbf{b}]$.
2. Use row operations to reduce the augmented matrix to row echelon form.
3. Write down the simplified system.
4. Backsolve.
   - Assign arbitrary values to the free variables.
   - Backsolve for the pivot variables.

Return

The Solution Set of $A\mathbf{x} = \mathbf{b}$

- The solution set is the set of all vectors that satisfy $A\mathbf{x} = \mathbf{b}$.
  - A solution set is best described by giving a parmetric presentation. This is provided automatically by the method of elimination and backsolving.
- The solution set for the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is called the *nullspace* of A, denoted by $\mathrm{null}(A)$.
  - A nullspace is best described by giving a basis.
- If $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$, the solution set for for the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is given by

$$\{\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h \,|\, \mathbf{x}_h \in \mathrm{null}(A)\}\,.$$

Basis of $\mathrm{null}(A)$

Definition: A set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ form a *basis* of $\mathrm{null}(A)$ if

1. $\mathrm{null}(A) = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$
2. $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are linearly independent.

- Bases are not unique, but every basis of $\mathrm{null}(A)$ has the same number of elements.
- The number of elements in a basis is the *dimension* of $\mathrm{null}(A)$.

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How Do We Know if $\mathbf{w} \in \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ as its columns.
2. Solve the system $M\mathbf{a} = \mathbf{w}$.
   a. If there are no solutions, $\mathbf{w}$ is *NOT* in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.
   b. If there is a solution $\mathbf{a} = (a_1, a_2, \cdots, a_k)^T$, then

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k$$

   is in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.

Return Product

When are $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ Linearly Independent?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ as its columns.
2. Find the nullspace, $\mathrm{null}(M)$.
   a. If $\mathrm{null}(M) = \{\mathbf{0}\}$, the vectors are linearly independent.
   b. If $\mathbf{a} \in \mathrm{null}(M)$, and $\mathbf{a} = (a_1, a_2, \cdots, a_k)^T \neq \mathbf{0}$, then

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k = \mathbf{0}$$

   and the vectors are linearly dependent.

Return Product

$n \times n$ Matrices

- An $n \times n$ matrix A is *invertible* if there is an $n \times n$ matrix B such that $AB = BA = I$. The matrix $A^{-1} = B$ is called the *inverse* of A.
- The $n \times n$ matrix A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side $\mathbf{b}$.

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Determinants

The determinant of an $n \times n$ matrix A can be computed using:

- Expansion by the i^{th} row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- Expansion by the j^{th} column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- Row or column operations to simplify the matrix.

More Properties

- If A has two equal rows, then $\det(A) = 0$.
- If A has a row of all zeros, then $\det(A) = 0$.
- If A has two equal columns, then $\det(A) = 0$.
- If A has a column of all zeros, then $\det(A) = 0$.

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Equivalent Properties

If A is an $n \times n$ matrix, the following are equivalent:

- $\det(A) \neq 0$.
- A is non-singular.
- A is invertible.
- The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side $\mathbf{b}$.
- $\mathrm{null}(A)$ is trivial, i.e., $\mathrm{null}(A) = \{\mathbf{0}\}$.
  - This means that the only solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$, the zero vector.

Systems of Differential Equations

- $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, where $\mathbf{x} \in \mathbf{R}^n$, $t \in I = (a, b)$.
- Initial value problem: $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ with $\mathbf{x}(t_0) = \mathbf{x}_0$
- Existence and uniqueness.
- Representation of solutions:
  - Component plots.
  - Curves in phase space. Parametric plots, $t \to \mathbf{x}(t)$.
- Reduction of higher order systems to first order systems.

Autonomous Systems

- $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.
- Uniqueness in phase space.
- $\mathbf{x}_0$ is an *equilibrium point* if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.
- $\mathbf{x}(t) = \mathbf{x}_0$ is the corresponding *equilibrium solution*.
- In phase space, an equilibrium solution plots as a point.
- Nullclines.

Homogeneous Linear Systems

- $\mathbf{x}' = A\mathbf{x}$, A is an $n \times n$ matrix.
- Solution strategy: look for a fundamental set of solutions, i.e., n linearly independent solutions.
- It suffices to find solutions to the initial value problems

$$\mathbf{x}_j' = A\mathbf{x}_j \quad \text{with} \quad \mathbf{x}_j(0) = \mathbf{v}_j$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a basis of $\mathbf{R}^n$.

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The Exponential of a Matrix

Definition: The *exponential* of the $n \times n$ matrix A is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_0^\infty \frac{1}{n!}A^n.$$

Theorem: The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is given by $\mathbf{x}(t) = e^{tA}\mathbf{v}$.

- It suffices to compute $\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j$ where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a basis of $\mathbf{R}^n$.

Method of Solution

Proposition: Suppose that A is an $n \times n$ matrix, λ is a number, and $\mathbf{v}$ is a vector.

1. If $[A - \lambda I]\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
2. If $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}(\mathbf{v} + t[A - \lambda I]\mathbf{v})$.
3. If $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = e^{\lambda t}\left(\mathbf{v} + t[A - \lambda I]\mathbf{v} + \frac{t^2}{2!}[A - \lambda I]^2\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v}\right).$$

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Eigenvalues and Eigenvectors

- λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$.
  - $p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of A.
- $\mathbf{v}$ is an eigenvector associated with the eigenvalue $\lambda \Leftrightarrow$ $\mathbf{v} \in \text{null}(A - \lambda I)$.
  - $\text{null}(A - \lambda I)$ is called the *eigenspace* of λ.

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Multiplicities and Generalized Eigenvectors

- The characteristic polynomial of A can be written as

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k},$$

  where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues.
  - The *algebraic multiplicity* of λ_j is q_j.
  - The *geometric multiplicity* of λ_j is d_j, the dimension of the eigenspace of $\lambda_j = \text{null}(A - \lambda_j I)$.
- $\mathbf{v}$ is a generalized eigenvector associated with λ_j if $[A - \lambda_j I]^p \mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$.
  - We can compute $\mathbf{x}(t) = e^{tA}\mathbf{v}$ if $\mathbf{v}$ is a generalzed eignevector.

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Procedure for Solving $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues and their algebraic multiplicities.
- For each eigenvalue λ with algebraic multiplicity q find q linearly independent solutions associated with λ:
  - Find the smallest integer p such that $\text{null}([A - \lambda I]^p)$ has dimension q.
  - Find a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ of $\text{null}([A - \lambda I]^p)$.
  - For $j = 1, 2, \ldots, q$ compute $\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j$.
  - Most of the time $q = 1$.
- This results in n linearly independent solutions.

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Procedure for a Complex Eigenvalue

- If λ is complex of algebraic multiplicity q. Then $\overline{\lambda}$ also has multiplicity q.
  - Find the smallest integer p such that $\mathrm{null}([A - \lambda I]^p)$ has dimension q.
  - Find a basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q$ of $\mathrm{null}([A - \lambda I]^p)$.
  - For $j = 1, 2, \ldots, q$ compute $\mathbf{z}_j(t) = e^{tA}\mathbf{w}$.
  - Compute $\mathbf{x}_j(t) = \mathrm{Re}(\mathbf{z}_j(t))$ and $\mathbf{y}_j(t) = \mathrm{Im}(\mathbf{z}_j(t))$.
- This results in $2q$ linearly independent real solutions corresponding to the eigenvalues λ and $\overline{\lambda}$.

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Planar Systems

- Distince real eigenvalues $\lambda_1 < \lambda_2$.
  - $\lambda_1 < 0 < \lambda_2$ — saddle point.
  - $\lambda_1 < \lambda_2 < 0$ — nodal sink.
  - $0 < \lambda_1 < \lambda_2$ — nodal source.
- Complex conjugate eigenvalues $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$.
  - $\alpha = \mathrm{Re}(\lambda) = 0$ — center.
  - $\alpha = \mathrm{Re}(\lambda) < 0$ — spiral sink.
  - $\alpha = \mathrm{Re}(\lambda) > 0$ — spiral source.
- The trace-determinant plane.

Product of a Matrix with a Vector

- The *product* of a matrix A and a vector $\mathbf{x}$ is the linear combination of the columns of A with the elements of $\mathbf{x}$ as coefficients.
- Example:

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 13 \\ -5 \\ 23 \end{pmatrix}$$
$$= 13 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (-5) \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 23 \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

Return