

Math 211

Review for the
Second Exam

November 9, 2003

Method of Solution for $A\mathbf{x} = \mathbf{b}$

There are four steps:

1. Use the augmented matrix $M = [A, \mathbf{b}]$.
2. Use row operations to reduce the augmented matrix to row echelon form.
3. Write down the simplified system.
4. Backsolve.
 - ◆ Assign arbitrary values to the free variables.
 - ◆ Backsolve for the pivot variables.

The Solution Set of $A\mathbf{x} = \mathbf{b}$

- The solution set is the set of all vectors that satisfy $A\mathbf{x} = \mathbf{b}$.
 - ◆ A solution set is best described by giving a parametric presentation. This is provided automatically by the method of **elimination and backsolving**.
- The solution set for the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is called the **nullspace** of A , denoted by $\text{null}(A)$.
 - ◆ A nullspace is best described by giving a basis.
- If \mathbf{x}_p is a particular solution of $A\mathbf{x} = \mathbf{b}$, the solution set for the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is given by

$$\{\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h \mid \mathbf{x}_h \in \text{null}(A)\}.$$

Basis of $\text{null}(A)$

Definition: A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ form a *basis* of $\text{null}(A)$ if

1. $\text{null}(A) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$
 2. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent.
- Bases are not unique, but every basis of $\text{null}(A)$ has the same number of elements.
 - The number of elements in a basis is the *dimension* of $\text{null}(A)$.

How Do We Know if $\mathbf{w} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k as its columns.
2. Solve the system $M\mathbf{a} = \mathbf{w}$.
 - a. If there are no solutions, \mathbf{w} is **NOT** in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$.
 - b. If there is a solution $\mathbf{a} = (a_1, a_2, \dots, a_k)^T$, then

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k$$

is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$.

When are $\mathbf{v}_1, \mathbf{v}_2, \dots,$ and \mathbf{v}_k Linearly Independent?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots,$ and \mathbf{v}_k as its columns.
2. Find the nullspace, $\text{null}(M)$.
 - a. If $\text{null}(M) = \{\mathbf{0}\}$, the vectors are linearly independent.
 - b. If $\mathbf{a} \in \text{null}(M)$, and $\mathbf{a} = (a_1, a_2, \dots, a_k)^T \neq \mathbf{0}$, then

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{0}$$

and the vectors are linearly dependent.

$n \times n$ Matrices

- An $n \times n$ matrix A is *invertible* if there is an $n \times n$ matrix B such that $AB = BA = I$. The matrix $A^{-1} = B$ is called the *inverse* of A .
- The $n \times n$ matrix A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side \mathbf{b} .

Determinants

The determinant of an $n \times n$ matrix A can be computed using:

- Expansion by the i^{th} row:

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- Expansion by the j^{th} column:

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- Row or column operations to simplify the matrix.

More Properties

- If A has two equal rows, then $\det(A) = 0$.
- If A has a row of all zeros, then $\det(A) = 0$.
- If A has two equal columns, then $\det(A) = 0$.
- If A has a column of all zeros, then $\det(A) = 0$.

Equivalent Properties

If A is an $n \times n$ matrix, the following are equivalent:

- $\det(A) \neq 0$.
- A is non-singular.
- A is invertible.
- The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side \mathbf{b} .
- $\text{null}(A)$ is trivial, i.e., $\text{null}(A) = \{\mathbf{0}\}$.
 - ♦ This means that the only solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$, the zero vector.

Systems of Differential Equations

- $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, where $\mathbf{x} \in \mathbf{R}^n$, $t \in I = (a, b)$.
- Initial value problem: $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ with $\mathbf{x}(t_0) = \mathbf{x}_0$
- Existence and uniqueness.
- Representation of solutions:
 - ◆ Component plots.
 - ◆ Curves in phase space. Parametric plots, $t \rightarrow \mathbf{x}(t)$.
- Reduction of higher order systems to first order systems.

Autonomous Systems

- $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.
- Uniqueness in phase space.
- \mathbf{x}_0 is an *equilibrium point* if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.
- $\mathbf{x}(t) = \mathbf{x}_0$ is the corresponding *equilibrium solution*.
- In phase space, an equilibrium solution plots as a point.
- Nullclines.

Homogeneous Linear Systems

- $\mathbf{x}' = A\mathbf{x}$, A is an $n \times n$ matrix.
- Solution strategy: look for a fundamental set of solutions, i.e., n linearly independent solutions.
- It suffices to find solutions to the initial value problems

$$\mathbf{x}'_j = A\mathbf{x}_j \quad \text{with} \quad \mathbf{x}_j(0) = \mathbf{v}_j$$

where $\mathbf{v}_1, \dots, \mathbf{v}_n$ are a basis of \mathbf{R}^n .

The Exponential of a Matrix

Definition: The *exponential* of the $n \times n$ matrix A is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots = \sum_0^{\infty} \frac{1}{n!}A^n.$$

Theorem: The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is given by $\mathbf{x}(t) = e^{tA}\mathbf{v}$.

- It **suffices** to compute $\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j$ where $\mathbf{v}_1, \dots, \mathbf{v}_n$ are a basis of \mathbf{R}^n .

Method of Solution

Proposition: Suppose that A is an $n \times n$ matrix, λ is a number, and \mathbf{v} is a vector.

1. If $[A - \lambda I]\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
2. If $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}(\mathbf{v} + t[A - \lambda I]\mathbf{v})$.
3. If $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = e^{\lambda t} \left(\mathbf{v} + t[A - \lambda I]\mathbf{v} + \frac{t^2}{2!}[A - \lambda I]^2\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v} \right).$$

Eigenvalues and Eigenvectors

- λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$.
 - ♦ $p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of A .
- \mathbf{v} is an eigenvector associated with the eigenvalue $\lambda \Leftrightarrow \mathbf{v} \in \text{null}(A - \lambda I)$.
 - ♦ $\text{null}(A - \lambda I)$ is called the *eigenspace* of λ .

Multiplicities and Generalized Eigenvectors

- The characteristic polynomial of A can be written as

$$p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k},$$

where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues.

- ♦ The *algebraic multiplicity* of λ_j is q_j .
- ♦ The *geometric multiplicity* of λ_j is d_j , the dimension of the eigenspace of $\lambda_j = \text{null}(A - \lambda_j I)$.
- \mathbf{v} is a generalized eigenvector associated with λ_j if $[A - \lambda_j I]^p \mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$.
 - ♦ We can **compute** $\mathbf{x}(t) = e^{tA} \mathbf{v}$ if \mathbf{v} is a generalized eigenvector.

Procedure for Solving $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues and their algebraic multiplicities.
- For each eigenvalue λ with algebraic multiplicity q find q linearly independent solutions associated with λ :
 - ♦ Find the smallest integer p such that $\text{null}([A - \lambda I]^p)$ has dimension q .
 - ♦ Find a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q$ of $\text{null}([A - \lambda I]^p)$.
 - ♦ For $j = 1, 2, \dots, q$ compute $\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j$.
 - ♦ Most of the time $q = 1$.
- This results in n linearly independent solutions.

Procedure for a Complex Eigenvalue

- If λ is complex of algebraic multiplicity q . Then $\bar{\lambda}$ also has multiplicity q .
 - ◆ Find the smallest integer p such that $\text{null}([A - \lambda I]^p)$ has dimension q .
 - ◆ Find a basis $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_q$ of $\text{null}([A - \lambda I]^p)$.
 - ◆ For $j = 1, 2, \dots, q$ **compute** $\mathbf{z}_j(t) = e^{tA}\mathbf{w}$.
 - ◆ Compute $\mathbf{x}_j(t) = \text{Re}(\mathbf{z}_j(t))$ and $\mathbf{y}_j(t) = \text{Im}(\mathbf{z}_j(t))$.
- This results in $2q$ linearly independent real solutions corresponding to the eigenvalues λ and $\bar{\lambda}$.

Planar Systems

- Distinct real eigenvalues $\lambda_1 < \lambda_2$.
 - ♦ $\lambda_1 < 0 < \lambda_2$ — saddle point.
 - ♦ $\lambda_1 < \lambda_2 < 0$ — nodal sink.
 - ♦ $0 < \lambda_1 < \lambda_2$ — nodal source.
- Complex conjugate eigenvalues $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$.
 - ♦ $\alpha = \operatorname{Re}(\lambda) = 0$ — center.
 - ♦ $\alpha = \operatorname{Re}(\lambda) < 0$ — spiral sink.
 - ♦ $\alpha = \operatorname{Re}(\lambda) > 0$ — spiral source.
- The trace-determinant plane.

Product of a Matrix with a Vector

- The *product* of a matrix A and a vector \mathbf{x} is the linear combination of the columns of A with the elements of \mathbf{x} as coefficients.
- Example:

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 13 \\ -5 \\ 23 \end{pmatrix} \\ = 13 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (-5) \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 23 \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$