

Math 211

Review for the Final Exam

December 7, 2003

The Final Exam

- The final will be comprehensive, covering material from the entire semester.
- The final will emphasize the material covered since the last exam.
- These slides will cover primarily the material covered since the last exam. They do *not* cover all of the material on the exam.
- Questions about any of the material of the course will be answered.

The Themes of the Course

- Modeling.
 - ◆ Population, finance, mixing, motion, vibrating spring, electrical circuits, ...
- Exact solutions.
 - ◆ **Separable** and **linear** equations in dimension 1.
 - ◆ Linear equations in higher dimension.
 - ▶ Matrix algebra.
 - ◆ Second order linear equations.
- Numerical solutions.
- Qualitative analysis.

Solving $\mathbf{x}' = A\mathbf{x}$

- A is an $n \times n$ matrix.
- Solution strategy: Look for a fundamental set of solutions, i.e., n linearly independent solutions.
- The function $\mathbf{x}(t) = e^{tA}\mathbf{v}$ solves the initial value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{v}$.
- Refined strategy: Compute $e^{tA}\mathbf{v}$ for n linearly independent vectors \mathbf{v} .
 - ◆ Computing $e^{tA}\mathbf{v}$ is hard except for specially chosen vectors \mathbf{v} .

Procedure for Solving $\mathbf{x}' = A\mathbf{x}$

- Find the **eigenvalues** of A and their algebraic **multiplicities**.
- For each eigenvalue λ with algebraic multiplicity q :
 - ♦ Find the smallest **integer** k for which $\text{null}([A - \lambda I]^k)$ has dimension q .
 - ♦ Find a **basis** for $\text{null}([A - \lambda I]^k)$.
 - ♦ For each vector \mathbf{v} in the basis **compute** the solution $\mathbf{x}(t) = e^{tA}\mathbf{v}$.
- The set of all of these solutions is a **fundamental** set of solutions.
- Replace complex solutions **with** real solutions.

Solutions to Higher Order Equations

Homogenous linear equation with constant coefficients:

$$y'' + py' + qy = 0$$

- Look for exponential solutions $y(t) = e^{\lambda t}$.
- *Characteristic polynomial:* $\lambda^2 + p\lambda + q$.
- If λ is a root of the characteristic polynomial then $y(t) = e^{\lambda t}$ is a solution.

Fundamental sets of solutions

- Two distinct real roots λ_1 and λ_2 :

$$y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}.$$

- One real root λ of multiplicity 2:

$$y_1(t) = e^{\lambda t} \quad \text{and} \quad y_2(t) = te^{\lambda t}.$$

- Complex conjugate roots $\lambda = \alpha \pm i\beta$:

$$y_1(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y_2(t) = e^{\alpha t} \sin \beta t.$$

Inhomogeneous Equations

$$y'' + Py' + Qy = f(t)$$

- If y_p is a particular solution, the general solution is

$$y(t) = y_p(t) + C_1y_1(t) + C_2y_2(t),$$

where y_1 and y_2 are a fundamental set of solutions to the homogeneous equation.

- The method of undetermined coefficients finds a particular solution $y_p(t)$.
 - ♦ If the forcing term $f(t)$ has a form which is replicated under differentiation, look for a particular solution of the same general form as the forcing term.

Cases

- If $f(t) = Ce^{bt}$, try $y_p(t) = ae^{bt}$.
- If $f(t) = A \cos \omega t + B \sin \omega t$, try $y_p(t) = a \cos \omega t + b \sin \omega t$.
 - ◆ Or try the complex method.
- If $f(t)$ is a polynomial of degree n , let y_p be a polynomial of degree n .
- Exceptional cases: Multiply expected form of y_p by t .
- Combination cases: Solve the equation in pieces.

Harmonic Motion

- Spring: $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$.
- Circuit: $I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t)$.
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2x = f(t).$$

- ◆ We call this the equation for *harmonic motion*.
- ω_0 is the *natural frequency*. c is the *damping constant*. $f(t)$ is the *forcing term*.

Unforced Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = 0$$

- Undamped: $c = 0$.
- Underdamped: $0 < c < \omega_0$.
- Critically damped: $c = \omega_0$.
- Over damped: $c > \omega_0$.

Forced Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

- A is the *forcing amplitude* and ω is the *forcing frequency*.
- The *general solution* is $x(t) = x_p(t) + x_h(t)$.
 - ♦ x_p is a *particular solution*. x_h is the *general solution* of the homogenous equation.
- Undamped: $c = 0$.
 - ♦ $\omega \neq \omega_0$: Beats.
 - ♦ $\omega = \omega_0$: Resonance.

Forced, Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

$$x(t) = x_p(t) + x_h(t).$$

- $c > 0$ implies that the roots have negative real part, so that $x_h(t) \rightarrow 0$ as t increases, so x_h is called the *transient term*.
- $x_p(t)$ is called the *steady-state solution*. It has the form

$$x_p(t) = G(\omega)A \cos(\omega t - \phi)$$

- ◆ x_p is oscillatory at the driving frequency.
- ◆ The amplitude of x_p is the product of the *gain*, $G(\omega)$, and the amplitude of the forcing function.
- ◆ x_p has a *phase shift* of ϕ with respect to the forcing function.

Qualitative Analysis

- Existence and uniqueness.
- For an autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, the basic question is, What happens to *all solutions* as $t \rightarrow \infty$?
- The easy cases: equilibrium points $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ and equilibrium solutions $\mathbf{x}(t) = \mathbf{x}_0$.
- Local qualitative analysis: What happens as $t \rightarrow \infty$ to all solutions that start near an equilibrium point \mathbf{x}_0 ?
 - ◆ This is the question of stability.
- Global qualitative analysis: What happens to *all solutions* as $t \rightarrow \infty$?

Stability

Suppose the autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has an equilibrium point at \mathbf{x}_0 .

- \mathbf{x}_0 is *stable* if every solution that starts close to \mathbf{x}_0 stays close to \mathbf{x}_0 .
- \mathbf{x}_0 is *asymptotically stable* if every solution that starts close to \mathbf{x}_0 stays near \mathbf{x}_0 and approaches \mathbf{x}_0 as $t \rightarrow \infty$.
 - ◆ \mathbf{x}_0 is called a *sink*.
- \mathbf{x}_0 is *unstable* if there are solutions starting arbitrarily close to \mathbf{x}_0 that move away from \mathbf{x}_0 .

Stability for $\mathbf{x}' = A\mathbf{x}$

- $D = 2$: Trace-determinant plane.
 - ◆ Generic equilibrium points: Nodal sources, spiral sources, nodal sinks, spiral sinks, saddles.
 - ◆ Nongeneric equilibrium points: Centers, etc.
- **Theorem:** Let A be an $n \times n$ real matrix.
 - ◆ Suppose the real part of every eigenvalue of A is negative. Then $\mathbf{0}$ is an **asymptotically stable** equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.
 - ◆ Suppose A has at least one eigenvalue with positive real part. Then $\mathbf{0}$ is an **unstable** equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.

Stability for $\mathbf{x}' = \mathbf{f}(\mathbf{x})$

- Suppose that \mathbf{x}_0 is an equilibrium point.
- The *linearization* at \mathbf{x}_0 is the system $\mathbf{u}' = J\mathbf{u}$, where J is the *Jacobian matrix* of \mathbf{f} at \mathbf{x}_0 .
- For the planar system $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$, the Jacobian is

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

Stability for $D = 2$

Theorem: Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the **linearization** at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Stability for $D \geq 1$

Theorem: Suppose that \mathbf{y}_0 is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of \mathbf{f} at \mathbf{y}_0 .

1. Suppose that the real part of every eigenvalue of J is negative. Then \mathbf{y}_0 is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then \mathbf{y}_0 is an unstable equilibrium point.

Global Qualitative Analysis

- What happens to *all solutions* as $t \rightarrow \infty$?
- The (forward) limit set of the solution $\mathbf{y}(t)$ that starts at \mathbf{y}_0 is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.
 - ♦ $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \rightarrow \infty$ such that $\mathbf{y}(t_k) \rightarrow \mathbf{x}$.
- What is $\omega(\mathbf{y}_0)$ for all \mathbf{y}_0 ?
 - ♦ What is the limit set for all solutions?
- In dimension 1, all limit sets are equilibrium points.

Limit Sets in Dimension 2

Theorem: If S is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then S is one of the following:

- An equilibrium point.
- A closed solution curve.
 - ◆ A closed solution curve is its own limit set.
 - ◆ The closed solution curve could be a limit cycle.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives*.

- In dimension 3 the answer is unknown.

Poincaré-Bendixson Theorem

Theorem: Suppose that R is a closed and bounded planar region that is positively invariant for a planar system. If R contains no equilibrium points, then there is a closed solution curve in R .

- The theorem is also true if the set R is negatively invariant.
- The closed solution curve might be a limit cycle.

Invariant Sets

Definition: A set S is *(positively) invariant* for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples include equilibrium points, and any solution curve.
- In dimension 2, invariant sets can frequently be found using:
 - ◆ nullclines,
 - ◆ polar coordinants.

Solving Separable Equations

$$\frac{dy}{dt} = g(y)h(t)$$

The three step solution process:

1. Separate the variables. $\frac{dy}{g(y)} = h(t) dt$ if $g(y) \neq 0$.
2. Integrate both sides. $\int \frac{dy}{g(y)} = \int h(t) dt$
3. Solve for $y(t)$.

Solving the Linear Equation $x' \equiv a(t)x + f(t)$

Four step process:

1. Rewrite as $x' - ax = f$.
2. Multiply by the integrating factor

$$u(t) = e^{-\int a(t) dt}.$$

Equation becomes $[ux]' = ux' - aux = uf$.

3. Integrate: $u(t)x(t) = \int u(t)f(t) dt + C$.
4. Solve for $x(t)$.

Keys to Computing $e^{tA}\mathbf{v}$

- If \mathbf{v} is a vector, we have

$$e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \frac{t^3}{3!}A^3\mathbf{v} + \dots$$

- Truncation:

- ◆ If $A\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = \mathbf{v}$.
- ◆ If $A^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v}$.
- ◆ If $A^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \dots + \frac{t^{k-1}}{(k-1)!}A^{k-1}\mathbf{v}.$$

- Law of Exponents: If $AB = BA$, then $e^{A+B} = e^A e^B$.
 - ◆ Implies that $e^{tA} = e^{\lambda t} e^{t[A-\lambda I]}$.

Proposition

Proposition: Suppose that A is an $n \times n$ matrix, λ is a number, and \mathbf{v} is a vector.

1. If $[A - \lambda I]\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
2. If $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}(\mathbf{v} + t[A - \lambda I]\mathbf{v})$.
3. If $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = e^{\lambda t} \left(\mathbf{v} + t[A - \lambda I]\mathbf{v} + \frac{t^2}{2!}[A - \lambda I]^2\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v} \right).$$

Eigenvalues and Eigenvectors

- λ is an *eigenvalue* of A if there is a nonzero vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$. If λ is an eigenvalue of A , then any vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with λ* .
- λ is an eigenvalue of $A \Leftrightarrow p(\lambda) = \det(A - \lambda I) = 0$. $p(\lambda)$ is the *characteristic polynomial* of A .
- \mathbf{v} is an eigenvector associated with the eigenvalue $\lambda \Leftrightarrow [A - \lambda I]\mathbf{v} = \mathbf{0}$. $\text{null}(A - \lambda I)$ is called the *eigenspace* of λ .
- If λ is an eigenvalue of A and $(A - \lambda I)^p \mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then \mathbf{v} is called a *generalized eigenvector* associated with λ . We can compute $e^{tA}\mathbf{v}$ for any generalized eigenvector.

Multiplicities

A an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_k$.

- The characteristic polynomial has the form

$$p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \cdot \dots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of λ_j is q_j .
 - ♦ $q_1 + q_2 + \dots + q_k = n$.
- The *geometric multiplicity* of λ_j is d_j , the dimension of the *eigenspace* of λ_j .
 - ♦ $1 \leq d_j \leq q_j$.
- There is an integer $k_j \leq q_j$ for which $\text{null}([A - \lambda_j I]^{k_j})$ has dimension q_j .

Replacing Complex Solutions with Real Solutions

- If A has complex eigenvalues, the **fundamental** set of solutions contains complex valued solutions.
- Complex solutions occur in complex conjugate pairs $\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t)$ and $\overline{\mathbf{z}(t)} = \mathbf{x}(t) - i\mathbf{y}(t)$.
- Replace $\mathbf{z}(t)$ and $\overline{\mathbf{z}(t)}$ with the real solutions $\mathbf{x}(t) = \text{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \text{Im}(\mathbf{z}(t))$.