Math 211

Lecture #3

Solutions to Differential Equations

August 29, 2003

Differential Equations

A *differential equation* is an equation involving an unknown function and one or more of its derivatives, in addition to the independent variable.

- Example: $y' = \frac{dy}{dt} = 2ty$
- General first order equation: $y' = \frac{dy}{dt} = f(t, y)$
- t is the *independent variable*.
- $y = y(t)$ is the *unknown function*.
- $y' = 2ty$ is of *order 1*.
- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is a *partial differential equation* of order 2.

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Solutions to Ordinary Differential Equations

The general first order equation can be written as

$$y' = f(t, y).$$

A *solution* is a function $y(t)$, defined for t in an interval, which is differentiable at each point and satisfies

$$y'(t) = f(t, y(t))$$

for every point t in the interval.

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Example: $y' = y + t$

- Is $y(t) = e^t - 1 - t$ a solution?
  - By substitution the left-hand side is

$$y'(t) = e^t - 1,$$

  - and the right-hand side is

$$y(t) + t = (e^t - 1 - t) + t = e^t - 1.$$

  - Since these are equal, $y(t) = e^t - 1 - t$ is a solution.
- Is $y(t) = e^t$ a solution ?
  - By substitution $y'(t) \neq y(t) + t$, so $y(t) = e^t$ is *not* a solution to the equation $y' = y + t$.

Verification by substitution is always available.

More about Solutions

- A solution is a function. What is a function?
  - An exact, algebraic formula (e.g., $y(t) = e^{t^2}$).
  - A convergent power series.
  - The limit of a sequence of functions.
- An ODE is a function generator.
- Two of the themes of the course are aimed at those solutions for which there is no exact formula.

An ODE is a Function Generator

Example: $y' = y^2 - t$, $y(0) = 0$

- There is no solution to this IVP which can be given using a formula.
- Nevertheless, there is a solution. We can find as many terms in the power series for $y(t)$ as we want.

$$y(t) = -\frac{1}{2}t^2 + \frac{1}{20}t^5 - \frac{1}{160}t^8 + \dots$$

Particular and General Solutions

For the equation $y' = 2ty$

- $y(t) = \frac{1}{2}e^{t^2}$ is a solution. It is a *particular solution*.
- $y(t) = Ce^{t^2}$ is a solution for any constant C. This is a *general solution*.

General solutions contain arbitrary constants. Particular solutions do not.

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Initial Value Problem (IVP)

A differential equation & an initial condition.

- Example: Find $y(t)$ with $y' = -2ty$ with $y(0) = 4$.
- General solution: $y(t) = Ce^{-t^2}$.
- Plug in the initial condition:

$$y(0) = 4,$$
$$Ce^0 = 4,$$
$$C = 4$$

Solution to the IVP: $y(t) = 4e^{-t^2}$.

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Normal Form of an Equation

The first order differential equation

$$y' = f(t, y)$$

is said to be in *normal form*.

- Example: The differential equation $(1 + t^2)y' + y^2 = t^3$ is not in normal form.
- Solve for y' to put the equation into normal form:

$$y' = \frac{t^3 - y^2}{1 + t^2}$$

- Many statements about differential equations require the equation to be in normal form.

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Interval of Existence

The largest interval over which a solution can exist.

- Example: $y' = -2ty$ with $y(0) = 4$.
  - The interval of existence is $\mathbf{R} = (-\infty, \infty)$.
- Example: $y' = 1 + y^2$ with $y(0) = 1$.
  - General solution: $y(t) = \tan(t + C)$
  - Initial Condition: $y(0) = 1 \Rightarrow y(t) = \tan(t + \pi/4)$
  - The solution exists and is continuous for $-\pi/2 < t + \pi/4 < \pi/2$.
  - The interval of existence is $-3\pi/4 < t < \pi/4$.

Geometric Interpretation of

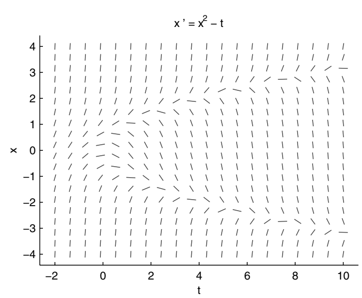
$$y' = f(t, y)$$

If $y(t)$ is a solution, and $y(t_0) = y_0$, then

$$y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0).$$

- The slope to the graph of $y(t)$ at the point (t_0, y_0) is given by $f(t_0, y_0)$.
- Imagine a small line segment attached to each point of the (t, y) plane with the slope $f(t, y)$.
- The result is called the *direction field* for the differential equation.

The Direction Field for $x' = x^2 - t$.



Autonomous Equations

- General equation: $\frac{dy}{dt} = f(t, y)$
- Autonomous equation: $\frac{dy}{dt} = f(y)$
- Examples:
  - $\frac{dy}{dt} = t - y^2$ is not autonomous.
  - $\frac{dy}{dt} = y^2 - 1$ is autonomous.

In an *autonomous equation* the right-hand side has no explicit dependence on the independent variable.

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Equilibrium Points

- An *equilibrium point* for the *autonomous equation* $\frac{dy}{dt} = f(y)$ is a point y_0 such that $f(y_0) = 0$.
- Corresponding to the equilibrium point y_0 there is the constant *equilibrium solution* $y(t) = y_0$.
- Example: $\frac{dy}{dt} = y(2 - y)/3$ is an autonomous equation.
  - The equilibrium points are $y_0 = 0$ or 2.
  - The corresponding equilibrium solutions are $y(t) = 0$ and $y(t) = 2$.

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Between Equilibrium Points

- $\frac{dy}{dt} = f(y) > 0 \Rightarrow\ y(t)$ is increasing.
- $\frac{dy}{dt} = f(y) < 0 \Rightarrow\ y(t)$ is decreasing.
- The graphs of solutions to first order equations cannot cross (uniqueness theorem).
- Example: $\frac{dy}{dt} = y(2 - y)/3$

Equilibrium point