Math 211

Lecture #4

Separable Equations

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Autonomous Equations

- General equation: $\dfrac{dy}{dt} = f(t, y)$
- Autonomous equation: $\dfrac{dy}{dt} = f(y)$
- Examples:
  - $\dfrac{dy}{dt} = t - y^2$ is not autonomous.
  - $\dfrac{dy}{dt} = y(1 - y)$ is autonomous.

In an *autonomous equation* the right-hand side has no explicit dependence on the independent variable.

Equilibrium Points

- An *equilibrium point* for the *autonomous equation* $\frac{dy}{dt} = f(y)$ is a point y_0 such that $f(y_0) = 0$.
- Corresponding to the equilibrium point y_0 there is the constant *equilibrium solution* $y(t) = y_0$.
- Example: $\frac{dy}{dt} = y(2-y)/3$ is an autonomous equation.
  - The equilibrium points are $y_1 = 0$ and $y_2 = 2$.
  - The corresponding equilibrium solutions are $y_1(t) = 0$ and $y_2(t) = 2$.

Between Equilibrium Points

- The graphs of solutions to first order equations cannot cross (uniqueness theorem).
- $\dfrac{dy}{dt} = f(y) > 0 \Rightarrow\ y(t)$ is increasing.
- $\dfrac{dy}{dt} = f(y) < 0 \Rightarrow\ y(t)$ is decreasing.
- Example: $\dfrac{dy}{dt} = y(2-y)/3$

Separable Equations

- General differential equation: $\dfrac{dy}{dt} = f(t, y)$
- Separable differential equation: $\dfrac{dy}{dt} = g(y)h(t)$
- In a *separable equation* the right-hand side is a product of a function $h(t)$ of the independent variable (t) and a function $g(y)$ of the unknown function (y).
- Examples:
  - $\dfrac{dy}{dt} = t - y^2$ is not separable.
  - $\dfrac{dy}{dt} = t \sec y$ is separable.
  - Any autonomous equation $y' = f(y)$ is separable.

Solving Separable Equations

Example: $y' = \frac{dy}{dt} = t \sec y$

- Step 1: Separate the variables:

$$\frac{dy}{\sec y} = t\,dt \quad \text{or} \quad \cos y\,dy = t\,dt$$

  - ♦ We have to worry about dividing by 0, but in this case $\sec y$ is never equal to 0.

- Step 2: Integrate both sides of $\cos y\,dy = t\,dt$

$$\int \cos y\,dy = \int t\,dt$$

$$\sin y + C_1 = \frac{1}{2}t^2 + C_2 \quad \text{or}$$

$$\sin(y(t)) = \frac{1}{2}t^2 + C$$

where $C = C_2 - C_1$.

- Step 3: Solve $\sin(y(t)) = \frac{1}{2}t^2 + C$ for $y(t)$
  - We get
    $$y(t) = \arcsin\left(C + \frac{1}{2}t^2\right).$$
  - This is the general solution to $\frac{dy}{dt} = t\sec y$.

Solving Separable Equations

$$\frac{dy}{dt} = g(y)h(t)$$

The three step solution process:

1. Separate the variables. $\dfrac{dy}{g(y)} = h(t)\,dt$ if $g(y) \neq 0$.
2. Integrate both sides. $\displaystyle\int \frac{dy}{g(y)} = \int h(t)\,dt$
3. Solve for $y(t)$.

Examples

- $y' = ry$ with $y(0) = -2, 0, 3$
- $y' = 2ty$ with $y(0) = -1, 0, 2$
- $R' = \dfrac{\sin t}{1+R}$ with $R(0) = 1, -2, -1$
- $x' = \dfrac{3t^2 x}{1+2x^2}$ with $x(0) = 1, 0$
- $y' = 1 + y^2$ with $y(0) = -1, 0, 1$

Why the Method Works

$$\frac{dy}{dt} = g(y)h(t)$$

$$\frac{1}{g(y)}\frac{dy}{dt} = h(t) \qquad \text{if } g(y) \neq 0$$

$$\int \frac{1}{g(y)}\frac{dy}{dt}\,dt = \int h(t)\,dt$$

$$\int \frac{1}{g(y)}\,dy = \int h(t)\,dt$$