Math 211

Lecture #5

Models of Motion

September 5, 2003

Models of Motion

History of models of planetary motion.

- Babylonians - 3000 years ago.
  - Initiated the systematic study of astronomy.
  - Collection of astronomical data.

Greeks

- Descriptive model - Ptolemy (~ 100).
  - Geocentric model.
  - Epicycles.
- Enabled predictions.
- Provided no causal explanation.
- This model was refined over the following 1400 years.

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Nicholas Copernicus (1543)

- Shifted the center of the universe to the sun.
- Fewer epicycles required.
- Still descriptive and provided no causal explanation.
- The shift to a sun centered universe was a major change in human understanding of their place in the universe.

Johann Kepler (1609)

- Based on experimental work of Tycho Brahe (1400).
- Three laws of planetary motion.
  1. Each planet moves in an ellipse with the sun at one focus.
  2. The line between the sun and a planet sweeps out equal areas in equal times.
  3. The ratio of the cube of the semi-major axis to the square of the period is the same for each planet.
- This model was still descriptive and not causal.

Isaac Newton

- Three major contributions.
  - Laws of mechanics.
    - Second law — $F = ma$.
  - Universal law of gravity.
  - Fundamental theorem of calculus.
    - $f' = g \Leftrightarrow \int g(x)\,dx = f(x) + C$.
    - Invention of calculus.
  - *Principia Mathematica* 1687

Isaac Newton (cont.)

- Laws of mechanics and gravitation were based on his own experiments and his understanding of the experiments of others.
- Derived Kepler's three laws of planetary motion.
- This was a causal explanation.
  - It works for any mechanical motion.
  - It is still used today.

Isaac Newton (cont.)

- *The Life of Isaac Newton* by Richard Westfall, Cambridge University Press 1993.
- Problems with Newton's theory.
  - The force of gravity was action at a distance.
  - Physical anomalies.
    - The Michelson-Morley experiment (1881-87).
  - Mathematical anomalies.

Albert Einstein

- Special theory of relativity – 1905.
- General theory of relativity – 1916.
  - Gravity is due to curvature of space-time.
  - Curvature of space-time is caused by mass.
  - Gravity is no longer action at a distance.
- All known anomalies explained.

Unified Theories

- Four fundamental forces.
  - Gravity, electromagnetism, strong nuclear, and weak nuclear.
- Last three can be unified by quantum mechanics. — Quantum chromodynamics.
- Currently there are attempts to include gravity.
  - String theory.
  - *The Elegant Universe : Superstrings, hidden dimensions, and the quest for the ultimate theory* by Brian Greene, W.W.Norton, New York 1999.

Return

The Modeling Process

- It is based on experiment and/or observation.
- It is iterative.
  - For motion we have ≥ 6 iterations.
  - After each change in the model it must be checked by further experimentation and observation.
- It is rare that a model captures all aspects of the phenomenon.

Linear Motion

- Motion in one dimension — $x(t)$ is the distance from a reference position.
  - Example: motion of a ball in the earth's gravity — $x(t)$ is the height of the ball above the surface of the earth.
- Velocity: $v = x'$. Acceleration: $a = v' = x''$.
- Newton's second law $F = ma$ becomes

$$x'' = F/m \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= F/m. \end{aligned}$$

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Motion of a Ball

- Acceleration due to gravity is (approximately) constant near the surface of the earth, so $F = -mg$, where $g = 9.8m/s^2$.
- Newton's second law becomes

$$x'' = -g \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -g. \end{aligned}$$

- Integrate the second equation: $v(t) = -gt + c_1$.
- Substitute into the first equation and integate: $x(t) = -\frac{1}{2}gt^2 + c_1 t + c_2$.

Air Resistance

Acts in the direction opposite to the velocity. Therefore

$$R(x, v) = -r(x, v)v \quad \text{where} \quad r(x, v) \geq 0.$$

There are many models. We will look at two different cases.

1. The resistance is proportional to velocity,

$$R = -rv.$$

2. The magnitude of the resistance is proportional to the square of the velocity,

$$R = -k|v|v.$$

$R = -rv$

- $R(x, v) = -rv$, r a positive constant. The total force is $F = -mg - rv$.
- Newton's second law becomes

$$mx'' = -mg - rv \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -\frac{mg + rv}{m}. \end{aligned}$$

- The solution to the second equation is

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}.$$

- Notice $\lim_{t\to\infty} v(t) = -\dfrac{mg}{r}$.
- The *terminal velocity* is $v_{\text{term}} = -\dfrac{mg}{r}$.

$R = -k|v|v$

- $R(x, v) = -k|v|v$, k a positive constant. The total force is $F = -mg - k|v|v$.
- Newton's second law becomes

$$mx'' = -mg - k|v|v \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -\frac{mg + k|v|v}{m}. \end{aligned}$$

- The equation for v is separable. However, the $|v|$ term means that we have to consider the cases $v > 0$ and $v < 0$ separately.

A Dropped Ball

- Suppose a ball is dropped from a high point. Then $v < 0$.
- The equation is $v' = \dfrac{-mg + kv^2}{m}$.
- The solution is

$$v(t) = \sqrt{\frac{mg}{k}} \frac{Ae^{-2t\sqrt{kg/m}} - 1}{Ae^{-2t\sqrt{kg/m}} + 1}.$$

- The terminal velocity is

$$v_{\text{term}} = -\sqrt{mg/k}.$$

Solving for $x(t)$

- Integrating $x' = v(t)$ is sometimes hard.
- Use the trick (see Exercise 2.3.7):

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

- If the acceleration is a function of the velocity only, the equation

$$v\frac{dv}{dx} = a$$

is separable.

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Problem

A ball is projected from the surface of the earth with velocity v_0. How high does it go?

- At $t = 0$, we have $x(0) = 0$ and $v(0) = v_0$.
- At the top we have $t = T$, $x(T) = x_{\max}$, and $v(T) = 0$.
- If $R = 0$, the acceleration is $a = -g$. The equation $v\dfrac{dv}{dx} = a$ becomes $v\,dv = -g\,dx$.
- Integrating we get $\int_{v_0}^{0} v\,dv = -\int_0^{x_{\max}} g\,dx$.
- Thus, $-\dfrac{v_0^2}{2} = -gx_{\max}$ or $x_{\max} = \dfrac{v_0^2}{2g}$.

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$R = -rv$

The acceleration is $a = -(mg + rv)/m$. The equation $v\dfrac{dv}{dx} = a$ becomes

$$\int_{v_0}^{0} \frac{v\,dv}{rv + mg} = -\int_0^{x_{\max}} \frac{dx}{m}.$$

Solving, we get

$$x_{\max} = \frac{m}{r}\left[v_0 - \frac{mg}{r}\ln\left(1 + \frac{rv_0}{mg}\right)\right].$$

Problem

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$R = -k|v|v$

Since $v > 0$, the acceleration is $a = -\dfrac{mg + kv^2}{m}$. The equation $v\dfrac{dv}{dx} = a$ becomes

$$\int_{v_0}^{0} \frac{v\,dv}{kv^2 + mg} = -\int_0^{x_{\max}} \frac{dx}{m}.$$

Solving, we get

$$x_{\max} = \frac{m}{2k}\ln\left(1 + \frac{kv_0^2}{mg}\right).$$

Problem