Math 211

Lecture #6

Linear Equations

September 8, 2003

Air Resistance

Acts in the direction opposite to the velocity. Therefore

$$R(x, v) = -r(x, v)v \quad \text{where} \quad r(x, v) \geq 0.$$

There are many models. We will look at three cases.

1. $R = 0$. $v' = -g$.
2. $R = -rv$, $v' = -\dfrac{mg + rv}{m}$.
3. $R = -k|v|v$, $v' = -\dfrac{mg + k|v|v}{m}$.

Return

Solving for $x(t)$

- Integrating $x' = v(t)$ is sometimes hard.
- Use the trick (see Exercise 2.3.7):

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

- The equation

$$v\frac{dv}{dx} = a = \frac{F}{m}$$

is usually separable.

Return Resistance

Problem

A ball is projected from the surface of the earth with velocity v_0. How high does it go?

- At $t = 0$, we have $x(0) = 0$ and $v(0) = v_0$.
- At the top we have $t = T$, $x(T) = x_{\max}$, and $v(T) = 0$.
- We have $v\dfrac{dv}{dx} = a = \dfrac{F}{m} = \dfrac{-mg + R}{m}$, so

$$\int_{v_0}^{0} \frac{v\,dv}{mg - R} = -\int_0^{x_{\max}} \frac{dx}{m}.$$

Cases

1. $R = 0.$ $\qquad x_{\max} = \dfrac{v_0^2}{2g}.$
2. $R = -rv.$

$$x_{\max} = \frac{m}{r}\left[v_0 - \frac{mg}{r}\ln\left(1 + \frac{rv_0}{mg}\right)\right].$$

3. $R = -k|v|v.$ $\qquad x_{\max} = \dfrac{m}{2k}\ln\left(1 + \dfrac{kv_0^2}{mg}\right).$

Linear Equations

A *linear equation* is one of the form

$$x' = a(t)x + f(t).$$

- Example: $x' = \tan(t)x + 3\sin^2(t)$
- The unknown function x and its derivative must appear *linearly*.
- The equation is *homogeneous* if $f = 0$
  - $x' = a(t)x$, e.g. $x' = \tan(t)x$
- The equation is *inhomogeneous* if $f \neq 0$

Homogenous Linear Equations

- Homogeneous linear equations are separable.

$$\frac{dx}{dt} = a(t)x$$
$$x(t) = Ae^{\int a(t)\,dt}$$

- Example: $x' = \tan(t)x$.

$$x(t) = Ae^{\sec^2 t}$$

Return

Example: $x' = 3x + e^{2t}$

- Step 1: Solve the associated homogeneous equation, $x_0' = 3x_0$.
  - A particular solution is $x_0(t) = e^{3t}$.
- Step 2: Try to find $v(t)$ such that $x = vx_0$ is a solution to the IHE. Substitute $x(t) = v(t)e^{3t}$ into the equation:

$$x'(t) = 3v(t)e^{3t} + v'(t)e^{3t}$$
$$3x + e^{2t} = 3v(t)e^{3t} + e^{2t}$$

  - $x = ve^{3t}$ is a solution if $v' = e^{-t}$, or $v(t) = -e^{-t} + C$.

Return

Example (cont.)

- Step 3: Write down the general solution:

$$\begin{aligned} x(t) &= v(t)x_0(t) \\ &= (-e^{-t} + C)e^{3t} \\ &= -e^{2t} + Ce^{3t}. \end{aligned}$$

Return Steps 1 & 2

The General Linear Equation $x' = a(t)x + f(t)$

- Step 1: Solve the associated homogeneous equation, $x_0' = ax_0$.
  - A particular solution is $x_0(t) = e^{\int a(t)\,dt}$.
- Step 2: Try to find $v(t)$ such that $x = vx_0$ is a solution to the IHE. Substitute $x = vx_0$ into the equation:

$$x' = vx_0' + v'x_0 = avx_0 + v'x_0$$
$$ax + f = avx_0 + f$$

  - $x = vx_0$ is a solution if $v' = f/x_0$, or

$$v(t) = \int \frac{f(t)\,dt}{x_0(t)} + C.$$

Return

Solving $x' = a(t)x + f(t)$ (cont.)

- Step 3: Write down the general solution:

$$\begin{aligned} x(t) &= v(t)x_0(t) \\ &= \left(\int \frac{f(t)\,dt}{x_0(t)}\right) x_0(t) + Cx_0(t), \end{aligned}$$

where

$$x_0(t) = e^{\int a(t)\,dt}.$$

Return Steps 1 & 2

Solution Procedure

To solve $x' = a(t)x + f(t)$:

1. Solve the homogeneous equation $x_0' = ax_0$.
2. Find v such that $x = vx_0$ is a solution by substituting into the equation.
3. Write down the general solution, $x(t) = v(t)x_0(t)$.

Return Steps 1 & 2 Step 3

Examples

- $x' = -4x + 8, \quad x(0) = 0.$
- $t\dfrac{dy}{dt} + 5y = 3t.$
- $x' = 2tx + e^{t^2}, \quad x(0) = 1.$
- $y' + y\cot(t) = 5e^{\cos(t)}.$
- $y' = 3y - t, \quad y(0) = 2.$
- $z' = (z+1)\cos t, \quad z(\pi) = -1.$

Return Formula Procedure

Solving Using an Integrating Factor

- This method is the first method explained in Section 2.4 of the textbook.
- The following slides motivate and explain the method.

Example: $x' = \tan(t)x + 3\sin^2(t)$

- Rewrite as $x' - \tan(t)x = 3\sin^2(t)$
- Multiply by $\cos t$.

$$\cos(t)x' - \sin(t)x = 3\sin^2(t)\cos(t)$$

The left hand side is the derivative of $\cos(t)x$. So

$$[\cos(t)x]' = 3\sin^2(t)\cos(t)$$

Return

- Integrate

$$\cos(t)x(t) = 3\int \sin^2(t)\cos(t)\,dt = \sin^3(t) + C$$

- Solve for x

$$x(t) = \frac{\sin^3(t) + C}{\cos(t)}.$$

How did we do that? Can we do it in general?

The Key Step for $x' = ax + f$

- Rewrite as $x' - ax = f$.
- Multiply by a function $u(t)$ so that

$$u[x' - ax] = [ux]'$$
$$ux' - aux = ux' + u'x$$

  - True if $u' = -au$. Linear, homogeneous

$$u(t) = e^{-\int a(t)\,dt} \quad \text{is one solution.}$$

  - u is called an *integrating factor*.

Solving the Linear Equation $x' = a(t)x + f(t)$

Four step process:

1. Rewrite as $x' - ax = f$.
2. Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\,dt}.$$

   Equation becomes $[ux]' = ux' - aux = uf$.
3. Integrate: $u(t)x(t) = \int u(t)f(t)\,dt + C$.
4. Solve for $x(t)$.