Math 211

Lecture #8

Existence & Uniqueness

September 12, 2003

Qualitative Analysis

- Do solutions always exist?
  - Do solutions to an initial value problem always exist?
- How many solutions are there?
  - How many solutions are there to an initial value problem?
- If we solve an IVP with an initial condition that is slightly wrong will the computed solution be close to the real one?
- Can we predict the behavior of solutions without having a formula?

Existence Theorem

Theorem: Suppose the function $f(t, y)$ is defined and continuous in the rectangle R in the ty-plane. Then given any point $(t_0, y_0) \in R$, the initial value problem

$$y' = f(t, y) \quad \text{with} \quad y(t_0) = y_0$$

has a solution $y(t)$ defined in an interval containing t_0. Furthermore the solution will be defined at least until the solution curve $t \longrightarrow (t, y(t))$ leaves the rectangle R.

Explanation of the Existence Theorem

- Hypotheses:
  - The equation is in normal form $y' = f(t, y)$.
  - The right hand side, $f(t, y)$, is continuous in the rectangle R.
  - The initial point (t_0, y_0) is in the rectangle R.
- Conclusions:
  - There is a solution starting at the initial point.
  - The solution is defined at least until the solution curve $t \longrightarrow (t, y(t))$ leaves the rectangle R.

Existence of a Solution

- The existence theorem does not guarantee an explicitly defined solution.
- In the proof, the solution is constructed as the limit of a sequence of explicitly defined functions.
- Frequently no explicit formula is possible.
- An ordinary differential equation is a function generator.

Interval of Existence

- Example: $y' = 1 + y^2 \quad \text{with} \quad y(0) = 0.$
- RHS $f(t, y) = 1 + y^2$ is defined and continuous on the whole ty-plane. The rectangle R can be any rectangle in the plane.
- Solution $y(t) = \tan t$ “blows up” at $t = \pm\pi/2$.
- Thus the size of the rectangle on which $f(t, y)$ is continuous does not say much about the interval of existence.

Uniqueness of Solutions

- How many solutions does an initial value problem have?
- The uniqueness of solutions to an initial value problem is the mathematical equivalent of being able to predict results in science and engineering.
- The uniqueness of solutions to a differential equation model is equivalent to a system being causal.

Example of Non-uniqueness

- Initial value problem

$$y' = y^{1/3} \quad \text{with} \quad y(0) = 0.$$

- The constant function $y_1(t) = 0$ is a solution.
- Solve by separation of variables to find that

$$y_2(t) = \begin{cases} \left(\dfrac{2t}{3}\right)^{3/2} & \text{, if } t > 0 \\ 0 & \text{, if } t \leq 0. \end{cases}$$

is also a solution.

Uniqueness Theorem

Theorem: Suppose the function $f(t, y)$ and its partial derivative $\partial f/\partial y$ are continuous in the rectangle R in the ty-plane. Suppose that $(t_0, x_0) \in R$. Suppose that

$$x' = f(t, x) \quad \text{and} \quad y' = f(t, y),$$

and that

$$x(t_0) = y(t_0) = x_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$x(t) = y(t).$$

Uniqueness Theorem

- Hypotheses:
  - The equation is in normal form $y' = f(t, y)$.
  - The right hand side, $f(t, y)$, and its derivative $\partial f/\partial y$ are continuous in the rectangle R.
  - The initial point (t_0, y_0) is in the rectangle R.
- Conclusions:
  - There is one and only one solution starting at the initial point.
  - The solution is defined at least until the solution curve $t \to (t, y(t))$ leaves the rectangle R.

Geometric Interpretation

- Solution curves cannot cross.
- They cannot even touch at one point.
- $y' = (y-1)(\cos t - y)$ and $y(0) = 2$. Show that $y(t) > 1$ for all t.
- $y' = y - (1-t)^2$ and $y(0) = 0$. Show that $y(t) < 1 + t^2$ for all t.

E & U for Linear Equations

Theorem: Suppose that $a(t)$ and $g(t)$ are continuous on an interval $I = (a, b)$. Then given $t_0 \in I$ and any y_0, the initial value problem

$$y' = a(t)y + g(t) \quad \text{with} \quad y(t_0) = y_0$$

has a unique solution $y(t)$ *which exists for all $t \in I$.*

- Notice that the RHS is

  $$f(t, y) = a(t)y + g(t), \quad \text{and} \quad \frac{\partial f}{\partial y} = a(t).$$

  These are continuous for $t \in I$ and all y.

DFIELD6

Get a geometric look at existence and uniqueness.

Theorem: Suppose $f(t,y)$, $\partial f/\partial y$ are continuous in the rectangle R. Let

$$M = \max_{(t,y)\in R} \left| \frac{\partial f}{\partial y}(t,y) \right|.$$

Suppose that (t_0, x_0) and (t_0, y_0) both lie in R, and

$$x' = f(t,x),\ x(t_0) = x_0 \ \&$$

$$y' = f(t,y),\ y(t_0) = y_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$|x(t) - y(t)| \leq |x_0 - y_0| e^{M|t-t_0|}.$$

Continuity in Initial Conditions

- Inequality: $|x(t) - y(t)| \le |x_0 - y_0| e^{M|t-t_0|}.$
- The good news:
  - By making sure that x_0 and y_0 are very close we can make the solutions $x(t)$ and $y(t)$ close for t in an interval containing t_0.
  - Solutions are *continuous in the initial conditions.*

Sensitivity with Respect to Initial Conditions

- Inequality: $|x(t) - y(t)| \le |x_0 - y_0| e^{M|t-t_0|}$.
- The bad news:
  - As $|t - t_0|$ increases the RHS grows exponentially.
  - Over long intervals in t the solutions can get very far apart. Solutions are *sensitive to initial conditions.*