

Math 211

Lecture #10

Population Models

September 17, 2003

Modeling Population

- Assume population changes are due to births and deaths only.
- Births are roughly proportional to population, $B = bP$
 - ◆ b is the *birth rate*. It is the average number of births per individual in one unit of time.
- Deaths are roughly proportional to population, $D = dP$.
 - ◆ d is the *death rate*. It is the probability that any one individual will die in one unit of time.
- Rate of change = births – deaths

$$\frac{dP}{dt} = B - D = bP - dP = rP$$

- ◆ $r = b - d$ is the *reproductive rate*.

The Malthusian Model

- In general, b and d , and therefore r , are not constants. They can depend on P , and perhaps also on t .
- If there exist sufficient resources in term of nutrients and space, b and d will be almost constant. Then the reproductive rate $r = b - d$ is almost a constant.
- If r is constant we have the *Malthusian model*.

$$\frac{dP}{dt} = rP \quad \text{with} \quad P(0) = P_0$$

- Solution: $P(t) = P_0 e^{rt}$
 - ♦ If $r = b - d > 0$, $P(t)$ grows exponentially.
 - ♦ If $r = b - d < 0$, $P(t)$ decays exponentially.

Evaluation of the Malthusian Model

Under what circumstances could the **Malthusian model** be a good model?

- Requires unlimited resources.
 - ◆ OK in laboratory experiments with small populations.
- Populations always outgrow the Malthusian model. This was the point that was made by Malthus.

The Logistic Model

- As the population increases individuals compete for resources — for food and for space.
- The **birth rate** b is the average number of births per individual in one unit of time.
 - ◆ As $P \nearrow$, $b \searrow$ because of competition.
 - ◆ Competition results from encounters.
 - ◆ The number of encounters by one individual is roughly proportional to P . \Rightarrow decrease in the birth rate is roughly proportional to P
 - ◆ Assume that $b = b_0 - b_1 P$
- Increase in the death rate d is $\sim P$
 - ◆ Assume that $d = d_0 + d_1 P$

The Logistic Model (cont.)

- The **reproductive rate** is

$$r = b - d = (b_0 - b_1P) - (d_0 + d_1P) = r_0 - r_1P$$

- The result is the **logistic model**

$$\begin{aligned}\frac{dP}{dt} &= rP \\ &= (r_0 - r_1P)P \\ &= r_0 \left(1 - \frac{P}{K}\right) P\end{aligned}$$

where $K = r_0/r_1$.

Qualitative Analysis of the Logistic Model

$$\frac{dP}{dt} = r_0 \left(1 - \frac{P}{K} \right) P$$

- Equation is autonomous.
- Equilibrium points are 0 & K .
- 0 is unstable, K is asymptotically stable.
- Any positive solution $P(t) \rightarrow K$ as $t \rightarrow \infty$.
 - ♦ K is the *carrying capacity*.
 - ♦ r_0 is the *reproductive rate at small populations*.

Solution of The Logistic Model

$$\frac{dP}{dt} = r \left(1 - \frac{P}{K}\right) P \quad \text{with} \quad P(0) = P_0$$

- Solution:

$$P(t) = \frac{K P_0}{P_0 + (K - P_0)e^{-rt}}$$

Estimating Parameters in the Malthusian Model

- Model: $P' = rP$

$$P(t) = P_0 e^{rt}$$

- ◆ Two parameters P_0 and r .
- ◆ Two measurements or observations needed to find the values of P_0 and r .
- ◆ It is better to use all of the data and use least squares (exponential or linear regression).
 - ▶ The MATLAB command is `polyfit`.

Estimating Parameters in the Logistic Model

- Logistic model $P' = r(1 - P/K)P$

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

- ◆ Three parameters, P_0 , r , and K .
- ◆ Three measurements or observations needed to find the values of P_0 , r , and K .
- ◆ It is better to use all of the data and use least squares. (Nonlinear regression)
 - ▶ The MATLAB command is `lsqnonlin`. This command is available in the Optimization Toolbox.

Modelling

- Two ways to write the rate of change of something, e.g., of a population P
 - ♦ The mathematical way is the derivative, $\frac{dP}{dt}$.
 - ♦ The **other way** involves scientific analysis.,

$$r \left(1 - \frac{P}{K} \right) P.$$

- Setting the two equal gives a differential equation model, in this case the logistic model

$$\frac{dP}{dt} = r \left(1 - \frac{P}{K} \right) P$$

Efficacy of the Logistic Model

- Does a very good job of modeling the growth of populations under controlled circumstances.
 - ◆ In laboratory experiments.
 - ◆ In other circumstances when the situation does not change.
- For human populations the model always breaks down.
 - ◆ Other factors become important, such as **immigration**, the advance of technology, and changing habits of life.