Math 211

Lecture #13

Runge-Kutta Methods

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Basic Problem

Numerically "solve" $y' = f(t, y)$ on the interval $[a, b]$ with $y(a) = y_0$.

- Find a discrete set of points

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b$$

- and values

$$y_0,\ y_1,\ y_2,\ \ldots,\ y_{N-1},\ y_N$$

  with y_j approximately equal to $y(t_j)$.

Runge-Kutta vs Euler

- Both use a fixed step size $h = (b - a)/N$.
- Euler's method
  - $y_k = y_{k-1} + f(t_{k-1}, y_{k-1}) \cdot h$
    - Uses one slope $f(t_{k-1}, y_{k-1})$
- Runge-Kutta methods
  - $y_k = y_{k-1} + S \cdot h$
    - S is a weighted average of two or more slopes.
    - Slopes chosen to increase the accuracy.

Return

Second Order Runge-Kutta

The basic RK step is $y_k = y_{k-1} + S \cdot h$

- RK2 uses $S = \frac{1}{2}(s_1 + s_2)$, where
    - ♦ $s_1 = f(t_{k-1}, y_{k-1})$
    - ♦ $s_2 = f(t_{k-1} + h, y_{k-1} + s_1 \cdot h)$
- $y_k = y_{k-1} + \frac{1}{2}(s_1 + s_2) \cdot h; \quad t_k = t_{k-1} + h$

Second Order Runge-Kutta – Algorithm

```
Input $t_0$ and $y_0$.
for $k = 1$ to $N$
    $s_1 = f(t_{k-1}, y_{k-1})$
    $s_2 = f(t_{k-1} + h, y_{k-1} + s_1 h)$
    $y_k = y_{k-1} + \frac{1}{2}(s_1 + s_2)\, h$
    $t_k = t_{k-1} + h$
```

2^{nd} Order R-K – Error Analysis

- The truncation error at each step is $\leq Ch^3$.
- There are $N = (b-a)/h$ steps, but truncation error can propagate exponentially.
- Computation shows that

$$\text{Max error} \leq C\left(e^{L(b-a)} - 1\right) h^2,$$

  where C & L are constants that depend on f.
- Good news: decreases like h^2 as h decreases.
- Bad news: can get exponentially large as $b-a$ increases.

Fourth Order Runge-Kutta

The basic RK step is $y_k = y_{k-1} + S \cdot h$

- RK4 uses $S = \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4)$, where
    - ◆ $s_1 = f(t_{k-1}, y_{k-1})$
    - ◆ $s_2 = f(t_{k-1} + h/2, y_{k-1} + s_1 \cdot h/2)$
    - ◆ $s_3 = f(t_{k-1} + h/2, y_{k-1} + s_2 \cdot h/2)$
    - ◆ $s_4 = f(t_{k-1} + h, y_{k-1} + s_3 \cdot h)$
- $y_k = y_{k-1} + \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4) \cdot h$

Fourth Order Runge-Kutta – Algorithm

```
Input $t_0$ and $y_0$.
for $k = 1$ to $N$
    $s_1 = f(t_{k-1}, y_{k-1})$
    $s_2 = f(t_{k-1} + h/2, y_{k-1} + s_1 \cdot h/2)$
    $s_3 = f(t_{k-1} + h/2, y_{k-1} + s_2 \cdot h/2)$
    $s_4 = f(t_{k-1} + h, y_{k-1} + s_3 \cdot h)$
    $y_k = y_{k-1} + \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4) \cdot h$
    $t_k = t_{k-1} + h$
```

4^{th} Order R-K – Error Analysis

- The truncation error at each step is $\leq Ch^5$.
- There are $N = (b-a)/h$ steps, but the truncation error can propagate exponentially.
- Computation shows that

$$\text{Max error} \leq C \left(e^{L(b-a)} - 1 \right) h^4,$$

where $C \,\&\, L$ are constants that depend on f.

- Good news: decreases like h^4 as h decreases.
- Bad news: can get exponentially large as $b-a$ increases.

MATLAB Routines `rk2.m` & `rk4.m`

- Syntax: `[t,y] = rk2(derfile,`$[t_0, t_f], y_0, h$`);`
  - ♦ `derfile` - derivative m-file defining the equation.
  - ♦ t_0 - initial time; t_f - final time.
  - ♦ y_0 - initial value.
  - ♦ h - step size.

Experimental Error Analysis

- IVP $y' = \cos(t)/(2y-2)$ with $y(0) = 3$
- Exact solution: $y(t) = 1 + \sqrt{4 + \sin t}$.
- For several step sizes solve using Runge-Kutta methods and compare with the exact solution.
- For several step sizes solve IVP using Euler's method and the Runge-Kutta methods and compare the errors with the 3 methods.
    - Use `odesolvedemo.m`.

Return

Euler's Method – Algorithm

```
Input $t_0$ and $y_0$.
for $k = 1$ to $N$
    $y_k = y_{k-1} + f(t_{k-1}, y_{k-1})h$
    $t_k = t_{k-1} + h$
```

Return

Error Analysis – Euler's method

- Truncation error at each step is $\leq Ch^2$.
- There are $N = (b-a)/h$ steps, but truncation error can grow exponentially.
- Computation shows that

  $$\text{Maximum error} \leq C \left(e^{L(b-a)} - 1 \right) h,$$

  where $C \,\&\, L$ are constants that depend on f.
- Good news: the error decreases as h decreases.
- Bad news: the error can get exponentially large as the length of the interval [i.e., b-a] increases.

Return