Math 211

Lecture #16

Geometry of Solution Sets

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Systems and Solutionos

- Consider the system

$$\begin{array}{r} 3x + 4y - 5z = 3 \\ -2x + 3z = -7 \end{array} \Leftrightarrow A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 3 & 4 & -5 \\ -2 & 0 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}.$$

- A *solution* is a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. $(2, -2, -1)^T$ is a solution.

Solution Sets

- The *solution set* for a system of equations is set of *all* solutions .
  - What kinds of sets can be solution sets?
  - Can a circle be the solution set for a system of linear equations?
- We will find all possibilities in 2 and 3 dimensions.
  - These will inform our intuition about higher dimensions.
- We will us geometry to find the answer.

One Equation in Two Variables

Example: $2x - 3y = 1$

- The solution set is a line in the plane.
- The solution set consists of all vectors of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ (-1+2x)/3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/3 \end{pmatrix} + x \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$$

- x is a free parameter, $-\infty < x < \infty$.

Parametric Equation for a Line

$$\mathbf{u} = \mathbf{u}_0 + x\mathbf{v}, \quad -\infty < x < \infty$$

- In our case $\mathbf{u}_0 = (0,\ -1/3)^T$ and $\mathbf{v} = (1,\ 2/3)^T$
- The vector $\mathbf{u}_0$ locates one point on the line.
- The vector $\mathbf{v}$ gives the direction of the line.
- The number x tells how far the point $\mathbf{u}$ is from $\mathbf{u}_0$.
  - The line extends infinitely far in the $\mathbf{v}$ and $-\mathbf{v}$ directions.

Two Equations in Two Variables

Example:

$$\begin{array}{r} 2x - 3y = 1 \\ x + y = 3 \end{array}, \quad \text{or} \quad \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- Two equations $\Rightarrow$ two lines.
- There are three possibilities. Two lines in the plane can:
  - intersect in one point.
    - In our example $(2,\ 1)^T$
  - be the same line, so the intersection in a line.
  - be parallel, so the intersection is empty.
    - Such systems are said to be *inconsistent*.

Possible Solution Sets in Dimension 2

There are precisely three possibilities:

- A line.
- A single point.
- The empty set, $\emptyset$. The system is inconsistent.

Can a circle be the solution set for a system of linear equations?

One Equation in Three Variables

Example: $2x - 3y + 4z = 1$

- The solution set is a plane in 3-space.
- The solution set is all vectors of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ (1-2x+3y)/4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + x \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 3/4 \end{pmatrix}$$

where x and y are free parameters ($-\infty < x, y < \infty$).

Parametric Equation for a Plane

$$\mathbf{u} = \mathbf{u}_0 + x\mathbf{v} + y\mathbf{w}, \quad -\infty < x, y < \infty$$

- In the example $\mathbf{u}_0 = (0,\ 0,\ 1/4)^T$, $\mathbf{v} = (1,\ 0,\ -1/2)^T$, and $\mathbf{w} = (0,\ 1,\ 3/4)^T$
- $\mathbf{u}_0$ locates one point on the plane.
- $\mathbf{v}$ and $\mathbf{w}$ give two different directions in the plane.
- $\mathbf{u}$ differs from $\mathbf{u}_0$ by the linear combination of $\mathbf{v}$ and $\mathbf{w}$ with coefficients x & y.
  - The plane extends infinitely far in any direction that is a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

Two Equations in Three Variables

Example:

$$\begin{aligned} 2x - 3y + 4z &= 1 \\ x + y - z &= 3 \end{aligned} \quad \text{or} \quad \begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- The solution set consists of all vectors of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 13 - 6x \\ 10 - 5x \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \\ 10 \end{pmatrix} + x \begin{pmatrix} 1 \\ -6 \\ -5 \end{pmatrix}$$

- This is a line in 3-space.

Three Equations in Three Variables

Example:

$$\begin{aligned} 2x - 3y + 4z &= 1 \\ x + y - z &= 3 \\ 3x - y + 3z &= 5 \end{aligned} \quad \text{or} \quad \begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

- The only solution is $(x, y, z)^T = (2,\ 1,\ 0)^T$.

Solution Sets in Dimension 3

- Two equations $\Rightarrow$ two planes
  - There are three possibilities — $\emptyset$, a line, or a plane.
- Three equations $\Rightarrow$ three planes
  - There are four possibilities — $\emptyset$, a point, a line, or a plane.
  - In the example the three planes intersect in one point.

Classification of Solution Sets in Dimension 3

There are precisely four possibilities:

- A plane.
- A line.
- A single point.
- The empty set, $\emptyset$. The system is inconsistent.

Can a sphere be a solution set of a system of linear equations?

Solution Sets in Higher Dimension

By analogy with dimensions 2 &3, we expect the following:

- The solution set could be $\emptyset$ if the system is inconsistent.
- The solution set could be a point.
- If a solution set contains 2 points, then it contains the line through them.
- If a solution set contains 3 points not on the same line, then it contains the plane though them.
- There will be a parametric representation for the solution set.

Solution Sets of Homogeneous Systems

- $\mathbf{0}$ is the vector with all entries $= 0$. $\mathbf{0}$ is referred to as the *zero vector* or the *origin.*
- A *homogeneous system* is one of the form $A\mathbf{x} = \mathbf{0}$.
- A homogeneous system always has $\mathbf{0}$ as a solution.
- Hence the solution set of a homogeneous system is never the empty set.