

Math 211

Lecture #17

Solving Systems of Equations

October 3, 2003

Solving Systems of Equations

- We want to find a way to find the solution set of any system.
- We will build towards the method by looking at a series of examples.
- We will start by solving a 2×2 system using three different, but closely related methods.

Example 1 — Elimination

$$x + y = 3$$

$$2x - 3y = 1$$

- **Method 1:** Solve the first equation for x and substitute into the second equation.
- **Method 2:** Add -2 times the first equation to the second equation to eliminate x .
- **Method 3:** Use the augmented matrix and add -2 times the first row to the second row.

Example 1 — Backsolving

- Using any of these methods we get the simpler system

$$x + y = 3$$

$$-5y = -5$$

- The simple system has the same solutions as the **original system**.
- The simple system is very easy to solve
 - ♦ Solve the last equation first, $y = 1$.
 - ♦ Then the first equation, $x = 2$.
 - ♦ This is called **backsolving**.

Method of Solution - 1st Edition

The method is called *elimination and backsolving*, or *Gaussian elimination*. There are four steps:

1. Write down the augmented matrix.
2. **Eliminate** as many coefficients as possible.
 - ◆ This is not well defined yet.
3. Write down the **simplified system**.
4. Solve the simplified system by backsolving.
 - ◆ We need to explore this further.

Elimination — Row Operations

- We only use operations on the **equations** which will lead to systems of equations with the same solutions. These are:
 - ◆ **Add** a multiple of one equation to another.
 - ◆ Interchange two equations.
 - ◆ Multiply an equation by a non-zero number.
- The corresponding operations on the rows of the augmented matrix are called **row operations**.
 - ◆ Add a multiple of one row to another.
 - ◆ Interchange two rows.
 - ◆ Multiply a row by a non-zero number.

Example 2

$$y - 2z = -1$$

$$2x - 3y + 4z = 5$$

$$-2z + 5y - 8z = -7$$

1. Write down the augmented matrix.

$$\left(\begin{array}{cccc} 0 & 1 & -2 & -1 \\ 2 & -3 & 4 & 5 \\ -2 & 5 & -8 & -7 \end{array} \right)$$

Example 2 — Elimination

2. Eliminate as many coefficients as possible.

♦ Interchange rows 1 and 2. $R_1 \leftrightarrow R_2$

♦ $R_3 \rightarrow R_3 + R_1$.

♦ $R_3 \rightarrow R_3 + (-2) \cdot R_2$. The result is

$$\begin{pmatrix} 2 & -3 & 4 & 5 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. Write down the simplified system.

$$2x - 3y + 4z = 5$$

$$y - 2z = -1$$

Example 2 — Backsolving

4. Solve the simplified system by backsolving.

$$2x - 3y + 4z = 5$$

$$y - 2z = -1$$

- ◆ z is a free variable. Set $z = t$. Solve second equation for $y = -1 + 2t$. Solve first equation for $x = 1 + t$.
- ◆ The solution set is a line in three space:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad -\infty < t < \infty.$$

The Goal of Elimination

- How **simple** can we make the augmented matrix?

$$\begin{pmatrix} P & * & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * & * \\ 0 & 0 & 0 & P & * & * & * & * & * \\ 0 & 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & P & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- P is a nonzero number, $*$ is any number.

Row Echelon Form

- The *pivot* of a row in a matrix is the first non-zero element from the left.
- A matrix is in *row echelon form* if every pivot lies strictly to the right of those in rows above.
- Any matrix can be reduced to row echelon form using the first two of the row operations.
- When an augmented matrix has been reduced to row echelon form, the corresponding system can be easily solved by backsolving.

Reduced Row Echelon Form

- Row echelon form, plus all pivots = 1 and all other entries in a pivot column are 0.

$$\begin{pmatrix} 1 & 0 & * & 0 & 0 & * & 0 & 0 & * \\ 0 & 1 & * & 0 & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 3

$$3x_2 - 4x_3 = -7$$

$$-x_1 + 2x_2 = -3$$

$$3x_1 + 2x_2 + x_3 = 2$$

1. The augmented matrix is

$$\left(\begin{array}{cccc} 0 & 3 & -4 & -7 \\ -1 & 2 & 0 & -3 \\ 3 & 2 & 1 & 2 \end{array} \right)$$

Example 3 — Elimination

2. Elimination:

◆ $R_1 \leftrightarrow R_2.$

◆ $R_3 \rightarrow R_3 + 3 \cdot R_1.$

◆ $R_3 \rightarrow R_3 + \left(-\frac{8}{3}\right) \cdot R_2.$

◆ $R_3 \rightarrow \frac{3}{35} \cdot R_3.$

- We get

$$\begin{pmatrix} -1 & 2 & 0 & -3 \\ 0 & 3 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Example 3 — Back Solving

3. The simplified system is

$$-x_1 + 2x_2 = -3$$

$$3x_2 - 4x_3 = -7$$

$$x_3 = 1$$

4. Backsolve: $x_3 = 1$, $x_2 = -1$, and $x_1 = 1$.
- The solution set is the single point $(1, -1, 1)$.

Elimination using MATLAB

- $R_i \rightarrow R_i + aR_j$
 - ◆ `>> M(i,:) = M(i,:) + a*M(j,:)`
- $R_i \leftrightarrow R_j$
 - ◆ `>> M([i,j],:) = M([j,i],:)`
- $R_i \rightarrow aR_i$
 - ◆ `>> M(i,:) = a*M(i,:)`

Example 4. $A\mathbf{x} = \mathbf{b}$

$$A = \begin{pmatrix} 1 & 2 & 5 & -1 \\ 1 & 2 & -3 & 8 \\ 3 & 6 & 7 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ -12 \\ -16 \end{pmatrix}$$

1. Augmented matrix:

$$M = [A, \mathbf{b}] = \begin{pmatrix} 1 & 2 & 5 & -1 & -2 \\ 1 & 2 & -3 & 8 & -12 \\ 3 & 6 & 7 & 6 & -16 \end{pmatrix}$$

Example 4 — Elimination

2. Elimination using MATLAB yields.

$$\begin{pmatrix} 1 & 2 & 5 & -1 & -2 \\ 0 & 0 & -8 & 9 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

♦ The yellow entries are the **pivots**.

Example 4 — Simplified System

3. Simplified system:

$$x_1 + 2x_2 + 5x_3 - x_4 = -2$$

$$-8x_3 + 9x_4 = -10$$

4. Backsolve:

- ◆ There are **pivots** in columns 1 & 3. These are *pivot columns*. x_1 and x_3 are called *pivot variables*.
- ◆ The other columns are called *free columns*. The variables x_2 and x_4 are called *free variables*.
- ◆ The free variables may be assigned arbitrary values: $x_2 = s$ and $x_4 = t$.

Example 4 — Backsolving

- ◆ **Backsolve** for the pivot variables.

$$x_3 = (10 + 9x_4)/8 = 5/4 + 9t/8$$

$$x_1 = -33/4 - 2s - 37t/8$$

- The solutions are the vectors

$$\mathbf{x} = \begin{pmatrix} -33/4 \\ 0 \\ 5/4 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -37/8 \\ 0 \\ 9/8 \\ 0 \end{pmatrix}$$

- The solution set is a plane in \mathbf{R}^4 .

Method of Solution for $A\mathbf{x} = \mathbf{b}$

There are **four steps**:

1. Use the augmented matrix $M = [A, \mathbf{b}]$.
2. Use **row operations** to reduce the augmented matrix to **row echelon form**.
3. Write down the simplified system.
4. Backsolve.
 - ◆ Assign arbitrary values to the **free variables**.
 - ◆ **Backsolve** for the pivot variables.