Math 211

Lecture #18

Properties of Solution Sets

October 6, 2003

Method of Solution for $A\mathbf{x} = \mathbf{b}$

The method is called *elimination and backsolving*, or *Gaussian elimination*. There are four steps:

1. Use the augmented matrix $M = [A,\ \mathbf{b}]$.
2. Use row operations to reduce the augmented matrix to row echelon form.
3. Write down the simplified system.
4. Backsolve.
   - Assign arbitrary values to the free variables.
   - Backsolve for the pivot variables.

Return

Elimination using MATLAB

- Row operations in MATLAB. `M = [A,b]`.
  - $R_i \to R_i + aR_j$
    - `>> M(i,:) = M(i,:) + a*M(j,:)`
  - $R_i \leftrightarrow R_j$
    - `>> M([i,j],:) = M([j,i],:)`
  - $R_i \to aR_i$
    - `>> M(i,:) = a*M(i,:)`
- The MATLAB command `rref`.

Return

Consistent Systems

- A system is *consistent* if it has solutions.
- Examples

$$A = \begin{pmatrix} -3 & 6 & 0 \\ -2 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad \mathbf{b}_1 = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \quad \mathbf{b}_2 = \begin{pmatrix} -9 \\ -6 \\ 7 \end{pmatrix}$$

  - The system $A\mathbf{y} = \mathbf{b}_1$ is inconsistent. The system $A\mathbf{y} = \mathbf{b}_2$ is consistent.
- A system is consistent if and only if the simplified system is consistent.
- The simplified system is consistent if and only if the row echelon form of the augmented matrix does not have a pivot in its last column.

Return Row echelon form

Homogeneous Systems

A homogeneous system has the form $A\mathbf{x} = \mathbf{0}$.

- The augmented matrix $M = [A,\ \mathbf{0}]$ has all zeros in the last column.
  - During elimination the column of zeros is unchanged.
  - It is not necessary to augment a homogeneous system.
- A homogeneous system is always consistent, since the zero vector $\mathbf{0}$ is always a solution.
- A homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nonzero solution if and only if the row echelon form of A has a free column.
- A homogeneous system of n equations and m unknowns with $n < m$ always has a nonzero solution.

Return

Example 1

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & 5 \end{pmatrix}$$

- The solution set to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t(-3, -2, 1)^T$ for $-\infty < t < \infty$.
  - This happens because the reduced echelon form of A has a free column.
- The system $A\mathbf{x} = (1, 1, 0)^T$ has no solutions.
  - The reduced echelon form of $M = [A, b]$ has a pivot in the last column.

Return

Example 2

$$B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & 4 \end{pmatrix}$$

- The reduced echelon form of B has no free columns.
- The solution set for the homogeneous system $B\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
- The inhomogeneous system $B\mathbf{x} = \mathbf{b}$ has a unique solution for any vector $\mathbf{b}$.

Square Homogenous Systems

Suppose that A is a square matrix.

- If the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nonzero solution then there are vectors $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ has no solutions.
- If the homogeneous system $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$.

Singular and Nonsingular Matrices

The $n \times n$ matrix A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is nonsingular if and only if the row echelon form of A has only nonzero entries along the diagonal.

- $\Leftrightarrow$ `rref(A) = I`.

Proposition: If the $n \times n$ matrix A is nonsingular then the equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for any vector $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.

- This is a result that we will use repeatedly.

Invertible Matrices

An $n \times n$ matrix A is *invertible* if there is an $n \times n$ matrix B such that $AB = BA = I$. The matrix B is called an *inverse* of A.

- If B_1 and B_2 are both inverses of A, then

$$B_1 = B_1(AB_2) = (B_1A)B_2 = B_2$$

- The inverse of A is denoted by A^{-1}.
- Invertible $\Rightarrow$ nonsingular.
- Nonsingular $\Rightarrow$ invertible.

Return

Computing the inverse A^{-1}

- Form the matrix $[A,\ I]$.
- Do elimination until the matrix has the form $[I,\ B]$.
- Then $A^{-1} = B$.
- Examples: $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & 4 \end{pmatrix}$
- In MATLAB use `inv`.

Structure of the Solution Set

Theorem: Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector $\mathbf{x}_h$ such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

- The solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution $\mathbf{x}_p$ and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.

Return

Row operations

The permissable operations on the rows of the augmented matrix are called *row operations*.

- Add a multiple of one row to another.
- Interchange two rows.
- Multiply a row by a non-zero number.

Return

Row Echelon Form

A matrix is in *row echelon form* if every pivot lies strictly to the right of those in rows above.

$$\begin{pmatrix} P & * & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * & * \\ 0 & 0 & 0 & P & * & * & * & * & * \\ 0 & 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & P & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- P is a pivot, $*$ is any number.

Return