Math 211

Lecture #19

Nullspaces and Subspaces

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Example of a Solution Set

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \\ 5 & 6 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$$

- The solution set of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}.$$

- The solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}.$$

The Solution Set of $A\mathbf{x} = \mathbf{b}$

Theorem: Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector $\mathbf{x}_h$ such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

- Thus, the solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution $\mathbf{x}_p$ and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.

Solution Set of a Homogeneous System

- The solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the *nullspace* of the matrix A. It is denoted by $\mathrm{null}(A)$. Thus

$$\mathrm{null}(A) = \{\mathbf{x} \,|\, A\mathbf{x} = \mathbf{0}\}\,.$$

- What are the properties of nullspaces?
- Is there a convenient way to describe them?

Properties of Nullspaces

Proposition: Let A be a matrix.

1. If $\mathbf{x}$ and $\mathbf{y}$ are in $\text{null}(A)$, then $\mathbf{x}+\mathbf{y}$ is in $\text{null}(A)$.
2. If a is a scalar and $\mathbf{x}$ is in $\text{null}(A)$, then $a\mathbf{x}$ is in $\text{null}(A)$.

Definition: A nonempty subset V of $\mathbf{R}^n$ that has the properties

1. if $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, $\mathbf{x}+\mathbf{y}$ is in V,
2. if a is a scalar, and $\mathbf{x}$ is in V, then $a\mathbf{x}$ is in V,

is called a *subspace* of $\mathbf{R}^n$.

Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through $\mathbf{0}$, $V = \{t\mathbf{v} \mid t \in \mathbf{R}\}$, is a subspace.
- A plane through $\mathbf{0}$, $V = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\}$, is a subspace.
- $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are subspaces of $\mathbf{R}^n$.
  - These are called the *trivial subspaces.*

Linear Combinations

Proposition: Any linear combination of vectors in a subspace V is also in V.

- Subspaces of $\mathbf{R}^n$ have the same linear structure as $\mathbf{R}^n$ itself.
- The nullspace of a matrix is a subspace, so it has the same linear structure as $\mathbf{R}^n$.
- The product of a matrix A and a vector $\mathbf{x}$ is the linear combination of the column vectors in A with the elements of $\mathbf{x}$ as coefficients.

Examples of Nullspaces

1.

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{null}(A) = \{a\mathbf{v} \,|\, a \in \mathbf{R}\}$, where $\mathbf{v} = (1, -1, 1)^T$.

2.

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \,|\, a, b \in \mathbf{R}\}$, where
$\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$.

- $\text{null}(B)$ consists of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$.

The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ is denoted by

$$\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

Proposition: If $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are all vectors in $\mathbf{R}^n$, then $V = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbf{R}^n$.

How do we know if $\mathbf{w}$ is in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$?

1. Form the matrix $V = [\mathbf{v}_1, \mathbf{v}_2, \cdots .\mathbf{v}_k]$ which has the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ as its columns.
2. Solve the system $V\mathbf{a} = \mathbf{w}$.
   a. If there are no solutions, $\mathbf{w}$ is *NOT* in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$?
   b. If there is a solution $\mathbf{a} = (a_1, a_2, \cdots, a_k)^T$, then

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k.$$

Examples

Let $\mathbf{v}_1 = (1,\, 2)^T$, $\mathbf{v}_2 = (1,\, 0)^T$, and $\mathbf{v}_3 = (2,\, 0)^T$.

- $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{R}^2$. (Proof?)
- $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_3) = \mathbf{R}^2$. (Proof?)
- $\mathrm{span}(\mathbf{v}_2, \mathbf{v}_3) = \mathrm{span}(\mathbf{v}_2)$. (Proof?)
  - $\mathrm{span}(\mathbf{v}_2, \mathbf{v}_3) = \{t\mathbf{v}_2 \,|\, t \in \mathbf{R}\}$.
  - $\mathbf{v}_2$ and $\mathbf{v}_3$ have the same direction.

Row operations

The permissible operations on the rows of the augmented matrix are called *row operations*.

- Add a multiple of one row to another.
- Interchange two rows.
- Multiply a row by a non-zero number.

Row Echelon Form

A matrix is in *row echelon form* if every pivot lies strictly to the right of those in rows above.

$$\begin{pmatrix} P & * & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * & * \\ 0 & 0 & 0 & P & * & * & * & * & * \\ 0 & 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & P & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- P is a pivot, $*$ is any number.

Return

Method of Solution for $A\mathbf{x} = \mathbf{b}$

The method is called *elimination and backsolving*, or *Gaussian elimination.* There are four steps:

1. Use the augmented matrix $M = [A,\ \mathbf{b}]$.
2. Use row operations to reduce the augmented matrix to row echelon form.
3. Write down the simplified system.
4. Backsolve.
   - Assign arbitrary values to the free variables.
   - Backsolve for the pivot variables.

Product of a Matrix with a Vector

- The *product* of a matrix A and a vector $\mathbf{x}$ is the linear combination of the columns of A with the elements of $\mathbf{x}$ as coefficients.
- Example:

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 13 \\ -5 \\ 23 \end{pmatrix}$$

$$= 13 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (-5) \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 23 \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$