

Math 211

Lecture #20

Basis of a Subspace

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The Nullspace of a Matrix

- The solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the *nullspace* of the matrix A . It is denoted by $\text{null}(A)$. Thus

$$\text{null}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}.$$

- What are the properties of nullspaces?
- Is there a convenient way to describe them?

Properties of Nullspaces

Proposition: Let A be a matrix.

1. If \mathbf{x} and \mathbf{y} are in $\text{null}(A)$, then $\mathbf{x} + \mathbf{y}$ is in $\text{null}(A)$.
2. If a is a scalar and \mathbf{x} is in $\text{null}(A)$, then $a\mathbf{x}$ is in $\text{null}(A)$.

Definition: A nonempty subset V of \mathbf{R}^n that has the properties

1. if \mathbf{x} and \mathbf{y} are vectors in V , then $\mathbf{x} + \mathbf{y}$ is in V ,
2. if a is a scalar, and \mathbf{x} is in V , then $a\mathbf{x}$ is in V ,

is called a *subspace* of \mathbf{R}^n .

Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through $\mathbf{0}$, $V = \{t\mathbf{v} \mid t \in \mathbf{R}\}$, is a subspace.
- A plane through $\mathbf{0}$, $V = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\}$, is a subspace.
- $\{\mathbf{0}\}$ and \mathbf{R}^n are subspaces of \mathbf{R}^n .
 - ♦ These are called the *trivial subspaces*.

The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).$$

Proposition: If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are all vectors in \mathbf{R}^n , then $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ is a **subspace** of \mathbf{R}^n .

How Do We Know if $\mathbf{w} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, and \mathbf{v}_k as its columns.
2. Solve the system $M\mathbf{a} = \mathbf{w}$.
 - a. If there are no solutions, \mathbf{w} is **NOT** in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$.
 - b. If there is a solution $\mathbf{a} = (a_1, a_2, \dots, a_k)^T$, then

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k$$

is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$.

Examples

Let $\mathbf{v}_1 = (1, 2)^T$, $\mathbf{v}_2 = (1, 0)^T$, and $\mathbf{v}_3 = (2, 0)^T$.

- $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{R}^2$. (Proof?)
- $\text{span}(\mathbf{v}_1, \mathbf{v}_3) = \mathbf{R}^2$. (Proof?)
- $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{v}_2)$. (Proof?)
 - ◆ $\text{span}(\mathbf{v}_2, \mathbf{v}_3) = \{t\mathbf{v}_2 \mid t \in \mathbf{R}\}$.
 - ◆ \mathbf{v}_2 and \mathbf{v}_3 have the same direction.

How do we find a minimal spanning set?

Linear Independence of Two Vectors

We need a condition that will keep **unnecessary vectors** out of a spanning list. We will work toward a general definition.

- Two vectors are *linearly dependent* if one is a scalar multiple of the other.
 - ♦ \mathbf{v}_2 and \mathbf{v}_3 are linearly dependent.
 - ♦ \mathbf{v}_1 and \mathbf{v}_2 are *linearly independent*.

Linear Independence of Three Vectors

- Three vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are *linearly dependent* if one is a linear combination of the other two.
 - ♦ Example: $\mathbf{v}_1 = (1, 0, 0)^T$, $\mathbf{v}_2 = (0, 1, 0)^T$, and $\mathbf{v}_3 = (1, 2, 0)^T$. Notice that

$$\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2, \quad \text{or} \quad \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

- Three vectors are *linearly dependent* if there is a non-trivial linear combination of them which equals the zero vector.
 - ♦ Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

Linear Independence

Definition: The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

When are $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ Linearly Independent?

1. Form the matrix $M = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ which has the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ as its columns.
2. Find the nullspace, $\text{null}(M)$.
 - a. If $\text{null}(M) = \{\mathbf{0}\}$, the vectors are linearly independent.
 - b. If $\mathbf{a} \in \text{null}(M)$, and $\mathbf{a} = (a_1, a_2, \dots, a_k)^T \neq \mathbf{0}$, then

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{0}$$

and the vectors are linearly dependent.

Example 1

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}$$

$$M = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} 1 & -1 & 5 \\ -2 & -3 & 0 \\ 0 & 2 & -4 \\ 2 & 0 & 6 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}(M)$ consists of all multiples of $\mathbf{a} = (-3, 2, 1)^T$.
- $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, so the vectors are **linearly dependent**.

Example 2

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

$$M = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} 1 & -1 & 5 \\ -2 & -3 & 0 \\ 0 & 2 & -4 \\ 2 & 0 & 3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = \{\mathbf{0}\}$.
- $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Basis of a Subspace

Definition: A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ form a *basis* of a subspace V if

1. $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$
2. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are **linearly independent**.

Example 1

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}(A) = \{a\mathbf{v} \mid a \in \mathbf{R}\}$, where $\mathbf{v} = (1, -1, 1)^T$.
- The vector $\mathbf{v} = (1, -1, 1)^T$ is a **basis** for $\text{null}(A)$.
 - ♦ $\text{null}(A)$ is the subspace of \mathbf{R}^3 with basis \mathbf{v} .

Example 2

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbf{R}\}$, where $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$.
- The vectors $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$ form a basis for $\text{null}(B)$.
 - ♦ $\text{null}(B)$ is the subspace of \mathbf{R}^4 with basis $\{\mathbf{v}, \mathbf{w}\}$.

Existence of a Basis

Proposition: Let V be a subspace of \mathbf{R}^n .

1. If $V \neq \{0\}$, then V has a basis.
2. Bases are not unique, but every basis of V has the same number of elements.

Definition: The *dimension* of a subspace V is the number of elements in a basis of V .

Example 3

$$A = \begin{pmatrix} 3 & -3 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 13 & -13 & 5 & -5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\text{null}(A)$ is the subspace of \mathbf{R}^4 with basis $(1, 1, 0, 0)^T$ and $(0, 0, 1, -1)^T$.
- $\text{null}(A)$ has dimension 2.
- In MATLAB, use commands `null(A)` or `null(A, 'r')`.

Example 4

$$A = \begin{pmatrix} 13 & -69 & 64 & -46 & 9 & -51 \\ -7 & 27 & -26 & 18 & -3 & 19 \\ -9 & 51 & -47 & 34 & -6 & 38 \\ 6 & -18 & 18 & -12 & 2 & -12 \\ 13 & -57 & 54 & -38 & 7 & -41 \\ -2 & 18 & -16 & 12 & -2 & 14 \end{pmatrix}$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & -5/6 & 2/3 & 0 & 5/6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 4 (cont.)

- $\text{null}(A)$ is the subspace of \mathbf{R}^6 with basis

$$\begin{pmatrix} -1/2 \\ 5/6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2/3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} -1/2 \\ -5/6 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

- $\text{null}(A)$ has dimension 3.

Product of a Matrix with a Vector

- The *product* of a matrix A and a vector \mathbf{x} is the linear combination of the columns of A with the elements of \mathbf{x} as coefficients.
- Example:

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 13 \\ -5 \\ 23 \end{pmatrix} \\ = 13 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (-5) \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 23 \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$