Math 211

Lecture #21

Determinants

October 15, 2003

Nonsingular Matrices

Let A be an $n \times n$ matrix. We know the following:

- A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side $\mathbf{b}$. (This is the definition.)
- If A is nonsingular then $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side $\mathbf{b}$.
- A is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.
  - $\mathrm{null}(A)$ is non-trivial $\Leftrightarrow$ A is singular.
- We need a way to determine if A is singular or nonsingular.

Determinants in 2D

- How do we decide if a matrix A is nonsingular?
- A is nonsingular if and only if when put into row echelon form, the matrix has nonzero entries along the diagonal.
- Example: the general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is nonsingular if and only if $ad - bc \neq 0$.

  - We define $ad - bc$ to be the *determinant* of A.

Determinants in 3D

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- The same (but more difficult) argument shows that A is nonsingular if and only if

$$\begin{aligned} & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ & \quad + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} \\ & \neq 0. \end{aligned}$$

- This will be the determinant of A.

Main Theorem

We will define the determinant of a square matrix A so that the next theorem is true.

Theorem: The $n \times n$ matrix A is nonsingular if and only if $\det(A) \neq 0$.

Corollary: If A is an $n \times n$ matrix, then $\mathrm{null}(A)$ contains a nonzero vector if and only if $\det(A) = 0$.

- The corollary contains the most important fact about determinants for ODEs.

Matrices and Minors

The general $n \times n$ matrix has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Definition: The *ij-minor* of an $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix A_{ij} obtained from A by deleting the i^{th} row and the j^{th} column.

Definition of Determinant

Definition: The *determinant* of an $n \times n$ matrix A is defined to be

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_{1j}).$$

- The definition is inductive.
  - It assumes we know how to compute the determinants of $(n-1) \times (n-1)$ matrices.
  - We start with the 2×2 matrix.

Example

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 3 & -2 & 4 \\ -1 & 5 & 3 \end{pmatrix}$$

$$= (-1)^2 \times 2 \times \det\begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix}$$

$$+ (-1)^3 \times 1 \times \det\begin{pmatrix} 3 & 4 \\ -1 & 3 \end{pmatrix}$$

$$+ (-1)^4 \times 0 \times \det\begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix}$$

$$= 2 \times (-26) - 13$$

$$= -65$$

Expansion by the i^{th} Row

For any i, we have

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the i^{th} row.*
- Example:

$$\det \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & 0 \\ 2 & -16 & 9 \end{pmatrix} = 4 \cdot \det \begin{pmatrix} 5 & 3 \\ 2 & 9 \end{pmatrix} = 156.$$

Properties of the Determinant

- The formula for the determinant of a matrix A is the sum of $n!$ products of the entries of A (sometimes $\times -1$.)
  - Each summand is the product of n entries, one from each row, and one from each column.
- The determinant of a triangular matrix is the product of the diagonal terms.
  - We can use row operations to compute determinants.

Row Operations and Determinants

If B is obtained from A by

- adding a multiple of one row to another, then

$$\det(B) = \det(A).$$

- interchanging two rows, then

$$\det(B) = -\det(A).$$

- multiplying a row by $c \neq 0$, then

$$\det(B) = c\det(A).$$

Example

$$A = \begin{pmatrix} -5 & 2 & 3 \\ 25 & -9 & -12 \\ 10 & 7 & 17 \end{pmatrix}$$

$$\det(A) = 50$$

More Properties

- If A has two equal rows, then $\det(A) = 0$.
- If A has a row of all zeros, then $\det(A) = 0$.
- $\det(A^T) = \det(A)$.
- If A has two equal columns, then $\det(A) = 0$.
- If A has a column of all zeros, then $\det(A) = 0$.

Column Operations and Determinants

If B is obtained from A by

- adding a multiple of one column to another, then

$$\det(B) = \det(A).$$

- interchanging two columns, then

$$\det(B) = -\det(A).$$

- multiplying a column by $c \neq 0$, then

$$\det(B) = c\det(A).$$

Expansion by a Column

We can also expand by a column.

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the j^{th} column.*

Example

$$A = \begin{pmatrix} -5 & -6 & 0 \\ 3 & 4 & 0 \\ -8 & -16 & 9 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= 9 \cdot \det \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \\ &= 9 \cdot (-2) \\ &= -18 \end{aligned}$$

Determinants and Bases

Proposition: A collection of n vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . ,$\mathbf{v}_n$ in $\mathbf{R}^n$ is a basis for $\mathbf{R}^n$ if and only if

$$\det([\mathbf{v}_1\,\mathbf{v}_2\,\ldots\,\mathbf{v}_n]) \neq 0.$$

Examples

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix} = 1.$$

$$\det \begin{pmatrix} 3 & -1 & 0 & 1 \\ 12 & -6 & 0 & 5 \\ 32 & -15 & -3 & 13 \\ 18 & -10 & -1 & 8 \end{pmatrix} = -1.$$