

Math 211

Lecture #22
Systems of ODEs

October 17, 2003

Predator-Prey Populations

- Consider a mixed population of predators (foxes) and prey (rabbits).
- The prey, $x(t)$, flourish in the absence of the predators.
- The predators, $y(t)$, depend on the prey as a food source, and would die out in the absence of the prey.
- For predation to take place there must be an encounter between a predator and a prey.

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Predator-Prey Model

The basic model is $x' = r_x \cdot x$ and $y' = r_y \cdot y$, where r_x and r_y are the reproductive rates.

- $r_x = a > 0$ if $y = 0$, and decreases as y increases.
 - ♦ $r_x = a - by$.
- $r_y = -c < 0$ if $x = 0$, and increases as x increases.
 - ♦ $r_y = -c + dx$.
- The system becomes: $x' = (a - by)x$
 $y' = (-c + dx)y$
 - ♦ This is called the *Lotka Volterra* model.
- MATLAB & pp1ane6.

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[Assumptions](#)

General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example 2:

$$x' = y$$

$$y' = -x$$

- Solution 1: $x_1(t) = \sin t$ and $y_1(t) = \cos t$
 - ♦ Verify by direct substitution.
- Solution 2: $x_2(t) = \cos t$ and $y_2(t) = -\sin t$
 - ♦ Verify by direct substitution.

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Predator Prey

General System in Higher D

$$x'_1 = f_1(t, x_1, x_2, \dots, x_n)$$

$$x'_2 = f_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots = \quad \vdots$$

$$x'_n = f_n(t, x_1, x_2, \dots, x_n)$$

- The *dimension* of a system is the number of unknown functions = the number of equations.
 - ♦ The predator-prey model has dimension 2.
- Example: A food chain.

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Planar system

Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$. Then

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad \text{and} \quad \mathbf{u}'(t) = \begin{pmatrix} u'_1(t) \\ u'_2(t) \end{pmatrix}.$$

- For the right-hand side, set

$$\mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} f(t, u_1, u_2) \\ g(t, u_1, u_2) \end{pmatrix}.$$

- Then

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

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2D Examples

- The predator-prey model system can be written

$$\mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} (a - bu_2)u_1 \\ (-c + bu_1)u_2 \end{pmatrix}.$$

- Example 2:

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}.$$

- These are *autonomous* systems.
 - The RHS has no explicit dependence on t .

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Vector Notation — General

- In higher dimensions, set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

- Example: A food chain.

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Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example 2:

$$\begin{aligned} x' &= y & \text{with} & & x(0) &= 2 \\ y' &= -x & & & y(0) &= 13 \end{aligned}$$

- PP model: Both the initial prey population and the initial predator population must be specified.

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Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

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Example of Reduction

- Third-order equation: $y''' + 2yy' = 3 \cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3 \cos t - 2x_1x_3$$

- This system is *not* autonomous.

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Geometric Interpretation of Solutions

- Parametric plot
 - ♦ Tangent vectors
- Vector fields
- Phase plane
- `pplane6` for planar autonomous systems.