Math 211

Lecture #22

Systems of ODEs

October 17, 2003

Predator-Prey Populations

- Consider a mixed population of predators (foxes) and prey (rabbits).
- The prey, $x(t)$, flourish in the absence of the predators.
- The predators, $y(t)$, depend on the prey as a food source, and would die out in the absence of the prey.
- For predation to take place there must be an encounter between a predator and a prey.

Predator-Prey Model

The basic model is $x' = r_x \cdot x$ and $y' = r_y \cdot y$, where r_x and r_y are the reproductive rates.

- $r_x = a > 0$ if $y = 0$, and decreases as y increases.
  - $r_x = a - by$.
- $r_y = -c < 0$ if $x = 0$, and increases as x increases.
  - $r_y = -c + dx$.
- The system becomes: $$\begin{aligned} x' &= (a - by)x \\ y' &= (-c + dx)y \end{aligned}$$
  - This is called the *Lotka Volterra* model.
- MATLAB & `pplane6`.

General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example 2:

$$x' = y$$

$$y' = -x$$

- Solution 1: $x_1(t) = \sin t$ and $y_1(t) = \cos t$
  - Verify by direct substitution.
- Solution 2: $x_2(t) = \cos t$ and $y_2(t) = -\sin t$
  - Verify by direct substitution.

General System in Higher D

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots &= \qquad \vdots \\ x_n' &= f_n(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

- The *dimension* of a system is the number of unknown functions = the number of equations.
  - The predator-prey model has dimension 2.
- Example: A food chain.

Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$. Then

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad \text{and} \quad \mathbf{u}'(t) = \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix}.$$

- For the right-hand side, set

$$\mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} f(t, u_1, u_2) \\ g(t, u_1, u_2) \end{pmatrix}.$$

- Then

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

2D Examples

- The predator-prey model system can be written

$$\mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} (a - bu_2)u_1 \\ (-c + bu_1)u_2 \end{pmatrix}.$$

- Example 2:

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \Leftrightarrow \quad \mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}.$$

- These are *autonomous* systems.
  - The RHS has no explicit dependence on t.

Vector Notation — General

- In higher dimensions, set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

- Example: A food chain.

Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example 2:

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \text{with} \quad \begin{aligned} x(0) &= 2 \\ y(0) &= 13 \end{aligned}$$

- PP model: Both the initial prey population and the initial predator population must be specified.

Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

Example of Reduction

- Third-order equation: $y''' + 2yy' = 3\cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = 3\cos t - 2x_1x_2$$

- This system is *not* autonomous.

Geometric Interpretation of Solutions

- Parametric plot
  - Tangent vectors
- Vector fields
- Phase plane
- `pplane6` for planar autonomous systems.